Hw 4

2 5

a) We can define the graph of a function as

$$G(f) = \{ (x, f(x)) \in \mathbb{R} \times [0, \infty) \}$$

We see that f is a series of points and thus if f is measurable, we can slice* the graph into many E_i, all of which will contain only a point and be zero sets. Thus by the zero slice theorem G will be a zero set.

* We slice on first dim, so each E_i contains a $(x, f(x))$.

b) No. From Tao 7.3 we know that non measurable sets in $\mathbb{R}$ exists.

Consider a non-measurable set $A \subset [0, \infty)$, let

$$f(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \qquad f: \mathbb{R} \mapsto [0, \infty)$$

Then $G(f)$ is a zero set but $f^{-1}([0,\infty))$ not measurable

d) By the stack exchange post from c), if we remove measurability we can generate a function that has positive outer measure, 0 measure (and thus isn't measurable) but can be sliced into infinitely many slices of measure 0. Thus measurability is a must.

e) Assume $G(f)$ has positive inner measure. Then exists $K \subset G(f)$ closed box with positive measure. Then $\exists\, x_1, x_2 \in K : f(x_1) = f(x_2),\ x_1 \neq x_2$. since box has positive sides. But $f(x)$ is unique for each x. Contradiction and $G(f)$ has zero inner measure.

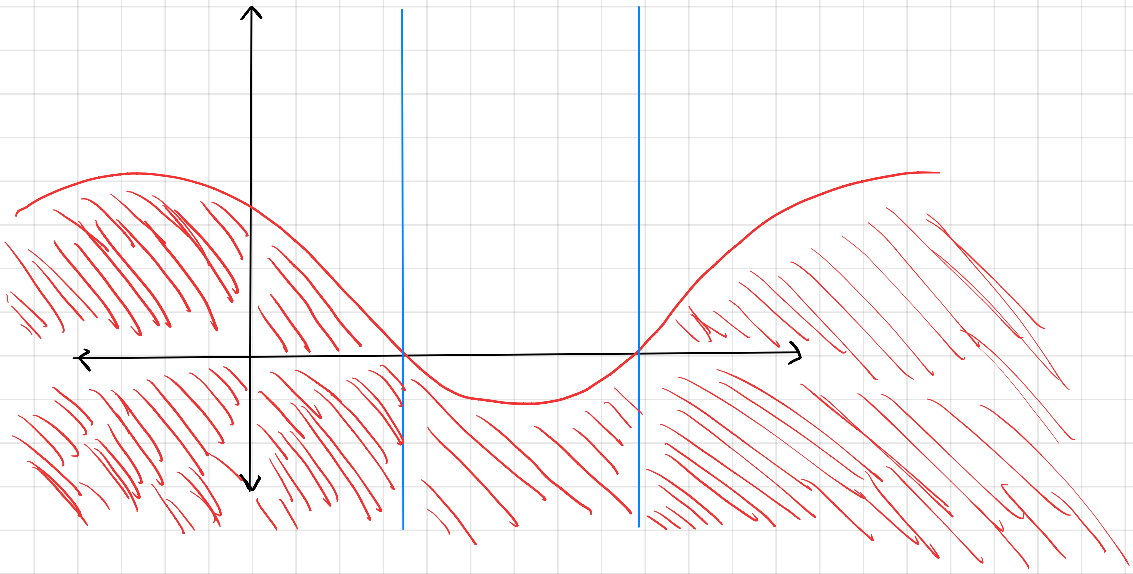
$\square$

f) The idea from c) is to generate a function from transfinite induction such that f_α estimates f down to a difference of a null set containing vertical lines. The complement of this, which is a G_δ set covering f, must then have infinite outer measure. Each of those partitions done by a vertical line has infinite measure. Uncountably infinite lines create uncountably infinite measures.

g) Due to a tight schedule this week i have not had time to answer this question. I will update the version on the course website with an answer when i have time.

28)

a)



Using this Picture let \\\ be the total undergraph.

Important is now that in order to find $m(\underline{U}(f))$ we must also include measure of the negative parts of f.

b)

From the hint we know from Exercise 23 that T is a mesomorphism and thus preserves measurability.

$T: (x,y) \Rightarrow (x, \frac{1}{y})$. We know that $U(f)$ is measurable, and for all $(x,y) \in U(f)$,* by def of $U(f)$ $y < f(x)$, so $\frac{1}{y} > \frac{1}{f(x)}$, which means $(x, \frac{1}{y}) \in (\hat{U}(f))^c =$ the "complete uppergraph"

Thus $T(Uf) = (\hat{U}\frac{1}{f})^c$. As T is a meseomorphism and by properties of measurable sets

Uf meas $\Leftrightarrow (\hat{U}\frac{1}{f})^c$ meas $\Leftrightarrow \hat{U}\frac{1}{f}$ meas $\Leftrightarrow U\frac{1}{f}$ meas $\square$

*

forgot case where $y=0$, although in that case $(x, +\infty) \in (\hat{U}f)^c$ so proof still holds.

c)

$T:(x,y)\mapsto(x,\log y)$ is a diffeomorphism $\Rightarrow$ meseomorphism.

Let $T^{-1}:(x,y)\mapsto(x,e^{y})$, which by same argument is a meseomorphism. $T(Uf)$ and $T(Ug)$ measurable.

We write T as $\log$ to ease notation:

$$
\begin{aligned}
\log(Uf),\ \log(Ug) \text{ meas } &\Leftrightarrow U(\log f),\ U(\log g) \text{ meas}\\
&\Leftrightarrow U(\log f)\cup U(\log g) \text{ meas}\\
&\Leftrightarrow U(\log(f\cdot g)) \text{ meas}\\
&\Leftrightarrow U(T(f\cdot g)) \text{ meas}\\
&\Leftrightarrow T^{-1}(U(T(f\cdot g))) \text{ meas}\\
&\Leftrightarrow U(T^{-1}(T(f\cdot g))) \text{ meas}\\
&= U(f\cdot g) \text{ meas}
\end{aligned}
$$

$\square$

d)

a)

In the removal of R, we now have $f: R^n \to R$

Thus $\underline{U}f = \{(x_1, \cdots, x_n, y) : y < f(x_1, \cdots x_n)\}$

Instead of Bothering with a plot depicting n+1 dimensions

Think of the set $\underline{U}f$ as being split into a

positive part and a negative part in the sense that

one has $f(x) \geq 0$ and the other $f(x) < 0$.

Let $\underline{U}^+f = \{(x_1, \cdots x_n, y) : y < f(x), f(x) \geq 0\}$ and

$\underline{U}^-f = \{(x_1, \cdots x_n, y) : y < f(x), f(x) < 0\}$, Then

Then $\underline{U}f = \underline{U}^-f \sqcup \underline{U}^+f$ and thus LHS meas (=) RHS meas

b)

Using the earlier proof but extend from x

to $(x_1, \cdots, x_n)$ works as well.

c)

Same case, extending to R^n does not change

Proof as only n+1 th dimension of map is touched.

e)

$* \, Uf = Uf^- \sqcup Uf^+$

if f and g have both signs, let $Uf^+ = Uf \cap (R \times [0, \infty))$ and

$Uf^- = Uf \cap (R \times (-\infty, 0))$. Same for g.* Then $\log f^-$ is not

defined, but $T: (x, y) \to (x, -y)$ is a meseomorphism and such

$U \log(T(f))$ is defined. if we use the method in c)

on uf^+ and ug^+, and on $T(uf^-)$ and $T(ug^-)$. Continue with

$T^{-1}(uf^-)$ and ug^+, and uf^+ and $T^{-1}(ug^-)$, for the final two use T^{-1} again in order to fix signs, all function combinations are measurable, and

$uf^+g^+ \sqcup uf^+g^- \sqcup uf^-g^+ \sqcup uf^-g^- = ufg$ is measurable.