Lec 1.

Math 105

How to read a text book:

think of the book as a big exercise book.

do every thing as practice problem.

lecture Notes

Lebesgue measure and integral

- Riemann integral over $\mathbb{R}$

$$\int_a^b f(x)\,dx = \text{area underneath the curve.}$$

At least: piecewise continuous functions are Riemann Integrable.

(in particular, piecewise constant functions)

shortcoming

- the underlying space is like $\mathbb{R}, \mathbb{R}^n$
  (the domain of Integration here is only bounded $[a,b]$)
- only bounded functions are considered. (e.g. $1/x$ cannot be integrated)
- If $f_n \to f$ pointwise, and f_n is Riemann Integrable, it's not true that f is Riemann Integrable
  (upgrade from pointwise $\to$ uniform convergence can preserve Integrability)

- Lebesgue Integral: $\int_\Omega f\,dx$ $\quad \Omega \subset \mathbb{R}^n$, $dx = dx_1 \dots dx_n$
  - what Ω do we allow? (Lebesgue Measurable set)
  - what f do we allow? (Lebesgue Integrable functions)

- Lebesgue Measure: $m(\Omega) = \int_\Omega 1 \cdot dx$

Intuitively (1) $m(\Omega)$ = area of Ω $\left(\int_a^b 1\,dy = b-a = |[a,b]| \right)$

(2) $\Omega \subset \mathbb{R}^2$, $m(\Omega)$ = area of Ω

(3) $\Omega \subset \mathbb{R}^3$, $m(\Omega)$ = volume

we know length of $|[a,b]| = b-a$

can we "consistently" define length (or general measure) for any subset of $\mathbb{R}^n$? A few desirable properties

① monotone: If $A \subset B \subset \mathbb{R}^n$ then $m(A) \leq m(B)$

② additivity: If $A \cap B = \phi$, then $m(A \cup B) = m(A) + m(B)$.

Question: If $A \subset B$, when will $m(A) = m(B)$

$B = [0,1]$, $A = [0,1)$, $m(A) = m(B) = 1$

$A = \{1\}$ $B = \{1,2\}$ in $\mathbb{R}$ $m(A) = m(B) = 0$

③ translation invariance: $\forall x \in \mathbb{R}^n$, $E \subset \mathbb{R}^n$

location shift doesn't matter

$m(E) = m(x+E)$ $\left(x+E = \{x+a \mid a \in E\} \right.$. e.g. $3+[1,2] = [3+1, 3+2] = [4,5]$ $\left.\right)$

Trouble: not possible to define such a measure on all subsets on $\mathbb{R}^n$

- Read Tao 7.3 for an example.
- Read wiki, Banach-Tarski paradox

a unit ball in $\mathbb{R}^3$ = disjoint union of finitely many pieces

→ after rotation and translation one can assemble them to 2 unit balls.

will be called "measurable"

cure: • Restrict the class of subsets in $\mathbb{R}^n$ to which we assign a measure,

Desired Properties / Axioms of measurable subsets (Tao-II Page 180)

Let M_n denote the set of measurable subsets in $\mathbb{R}^n$.

notation if S is a set, 2^S denote the set of subsets in S

we want

(1) If $\mathcal{U} \subset \mathbb{R}^n$ is open, then $\mathcal{U} \in M_n$ ($\mathcal{U}$ is measurable)

(2) If $\mathcal{U} \in M_n$, then $\mathcal{U}^c = \mathbb{R}^n \setminus \mathcal{U}$ is also in M_n

(3) If $\mathcal{U}, V \in M_n$, then $\mathcal{U} \cap V$ and $\mathcal{U} \cup V$ are measurable.

(2)(3) implies M_n is a Boolean algebra