Lec.2.

1 outer measure of any subset in $\mathbb{R}^n$

- Def $\forall E \subset \mathbb{R}^n$

$$m^*(E) = \inf\left\{ \sum_{i=1}^{\infty} vol(B_i) \,,\ \{B_i\} \text{ is an countable open cover of } E \text{ by boxes} \right\}$$

- Lemma 7.2.5 (xiii)
  - (v) no box is needed for cover.
  - (vi) by definition inf over nonegative numbers
- pf: (vii). For any open cover $\{B_i\}$ of B, it is also an open cover of A. And if $M, N \subset \mathbb{R}$ $M \supset N$ then $\inf M \le \inf N$

  Thus $m^*(A) \le m^*(B)$ ∎

- (viii) Finite sub-additivity W.T.S $m^*(A \cup B) \le m^*(A) + m^*(B)$

Try proving $m^*(A) + m^*(B) \ge$ (Total Area of some covering of A. and covering of B) $- \varepsilon$

then (total area) $\ge m^*(A \cup B)$

thus $\forall \varepsilon > 0$, $m^*(A) + m^*(B) \ge m^*(A \cup B) - \varepsilon$

$\Rightarrow m^*(A) + m^*(B) \ge m^*(A \cup B)$

$\because m^*(A) = \inf\{\sum Vol(B_i) \mid \{B_i\} \text{ cover } A\}$

$\therefore \forall \varepsilon > 0 \quad \exists$ covering $\{B_i\}$ s.t. $\sum |B_i| \le m^*(A) + \varepsilon$

Similarly, do it for B., Then take the union of the 2 countable covers to get $A \cup B$.

- (x) W.T.S $\forall \varepsilon > 0$, $\exists$ a collection of open covers $\{B_i^{(j)}\}$ for A_j s.t.

$$m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_{j}^{\infty} m^*(A_j) + \varepsilon$$

$$= \sum_{j=1}^{\infty} \left(m^*(A_j) + \varepsilon/2^j\right)$$

we can find open cover $\{B_i^{(j)}\}$ for A_j s.t.

$$m^*(A_j) + \varepsilon/2^j \ge \sum_{i=1}^{\infty} |B_i^{(j)}|$$

And $\displaystyle\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} |B_i^{(j)}|\right) \ge m^*\left(\bigcup_{j=1}^{\infty} A_j\right)$

countable

i.e a set with 3 operations ① NOT $(\ )^c$ ② AND $(\ )\cap(\ )$ ③ OR $(\ )\cup(\ )$

(4) We want M_n to be a σ-algebra, namely, If $U_1, U_2, \dots$ is a sequence of measurable sets. then $\bigcup_n U_n$ is measurable $\bigcap_n U_n$ is measurable.

$m_n : M_n \to \mathbb{R}_{\geq 0}$

Axioms for Lebesgue Measures (read Tao)

Thm: There exists a definition of M_n and Lebesgue Measure satisfying all the axioms.

Definition (outer measure): $\forall E \subset \mathbb{R}^n$.

$$m^*(E) = \inf\left\{ \sum_{i=1}^{\infty} \mathrm{vol}(B_i) ,\ \bigcup_{i=1}^{\infty} B_i \supset E,\ B_i \subset \mathbb{R}^n \text{ are open boxes}\right\}$$

open box in $\mathbb{R}^1$: (a, b)

$\mathbb{R}^2$: $(a_1, b_1) \times (a_2, b_2)$

$\mathrm{vol}(B) = \prod_{j=1}^{n} (b_j - a_j)$

$B = \prod_{j=1}^{n} (a_j, b_j)$

- outer measure is defined for ALL subsets in $\mathbb{R}$

In discussion: try to prove properties of $m^*(E)$

Lemma 7.2.5 7.2.6 $m^*(\text{box}) = \mathrm{vol}(\text{box})$

prep 7.2.6.

Recall : · compact set in $\mathbb{R}^n$ $\Longleftrightarrow$ closed and bounded

· Riemann integral

$$\mathrm{vol}([a,b]) = b-a = \int_a^b 1\,dx = \int_{\mathbb{R}} \underset{\substack{\uparrow \\ \text{indicator}}}{1_{[a,b]}}\,dx$$

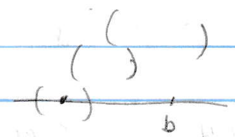
n dim $\mathrm{vol}([a_1,b_1]\times[a_2,b_2]\times\dots\times[a_n,b_n])$

$$= \int_{\mathbb{R}^n} 1_B(x) = dx$$

$dx = dx_1 dx_2 \dots dx_n$

same is true for open boxes

pf: ($n=1$ case) because $B=[a,b]$ is compact, hence any open cover of B can be reduced to a finite subcover. Let $\{B_i\}_{i=1}^N$ be a finite open cover. of B.



WTS

$$\sum_{i=1}^N |B_i| \geq \mathrm{vol}(B)$$

let $f_i(x) = 1_{B_i}(x)$, then $\sum_{i=1}^N |B_i| = \sum_{i=1}^N \left(\int_{\mathbb{R}} f_i dx\right)$

$$= \int_{\mathbb{R}} \sum_{i=1}^N f_i\,dx$$

claim $f(x) \geq 1_B(x)$ indeed $B \subset \bigcup_{i=1}^N B_i$

thus $1_B \leq 1_{B_i}$

($n=2$) case WTS given any finite cover $\{B_i\}_{i=1}^N$ of B.

that $\sum_{i=1}^N |B_i| \geq |B|$

again $|B_i| = \int_{\mathbb{R}^2} 1_{B_i}(x_1,x_2)\,dx_1 dx_2 = \int_{\mathbb{R}} w_i \cdot 1_{B_{i1}}(x_1)\,dx_1$.

going to integrate along x_2

$$\sum_{i=1}^N \int 1_{B_i}(x)\,dx_1 dx_2 = \int \sum_{i=1}^N 1_{B_i}(x)\,dx_1 dx_2 =$$

claim: $f(x_1) \geq 1_{[a_1,b_1]}(x_1) \cdot |b_2 - a_2|$ this follows by induction hypothesis (the $n=1$ case) applied to the line with given x_1