prep 7.2.6.

Recall : · compact set in $\mathbb{R}^n$ $\Longleftrightarrow$ closed and bounded

· Riemann integral

$$\mathrm{vol}([a,b]) = b-a = \int_a^b 1\,dx = \int_{\mathbb{R}} 1_{[a,b]}\,dx$$

indicator

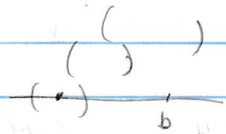
n dim $\mathrm{vol}([a_1,b_1]\times[a_2,b_2]\times\ldots\times[a_n,b_n])$

$$= \int_{\mathbb{R}^n} 1_B(x) = dx$$

$$dx = dx_1 dx_2, \ldots dx_n$$

same is true for open boxes

pf : ($n=1$ case) because $B=[a,b]$ is compact, hence any open cover of B can be reduced to a finite subcover. Let $\{B_i\}_{i=1}^{N}$ be a finite open cover of B.



WTS

$$\sum_{i=1}^{N} |B_i| \geq \mathrm{vol}(B)$$

let $f_i(x) = 1_{B_i}(x)$, then $\sum_{i=1}^{N} |B_i| = \sum_{i=1}^{N}\left(\int_{\mathbb{R}} f_i\,dx\right)$

$$= \int_{\mathbb{R}} \sum_{i=1}^{N} f_i\,dx$$

claim $f(x) \geq 1_B(x)$ indeed $B \subset \bigcup_{i=1}^{N} B_i$

thus $1_B \leq 1_{B_i}$

($n=2$) case WTS given any finite cover $\{B_i\}_{i=1}^{N}$ of B. that $\sum_{i=1}^{N} |B_i| \geq |B|$

again $|B_i| = \int_{\mathbb{R}^2} 1_{B_i}(x_1,x_2)\,dx_1 dy_2 = \int_{\mathbb{R}} w_i \cdot 1_{B_{i1}}(x_1)\,dx_1$

going to integrate along y_2

$$\sum_{i=1}^{N} \int 1_{B_i}(x)\,dx_1 dy_2 = \int \sum_{i=1}^{N} 1_{B_i}(x)\,dx_1 dy_2 =$$

claim: $f(x_1) \geq 1_{[a_1,b_1]}(x_1) \cdot |b_2 - a_2|$ this follows by induction hypothesis (the $n=1$ case) applied to the line with given x_1

Lec 3. Measurable Sets

Recall: • $m^*(A) := \inf\{\sum vol(B_i),\ \{B_i\}$ is countable collection of open boxes covering $A\}$

• properties
  • $m^*(\emptyset) = 0$
  • $A \subset B \Rightarrow m^*(A) \le m^*(B)$ monotonicity
  • countable subadditivity $A = \bigcup_{i=1}^{\infty} A_i$
    then $m^*(A) \le \sum_{i=1}^{\infty} m^*(A_i)$

• Def (measurable sets): $E \subset \mathbb{R}^n$ is measurable iff $\forall A \subset \mathbb{R}^n$ $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$

Today:

Lemma 7.4.2. (half spaces in $\mathbb{R}^n$ are measurable)
i.e. $\{(x_1, \dots, x_n) \mid x_n > 0\}$ is measurable in $\mathbb{R}^n$

pf. for $(n=1)$: We want to show, $\forall A \subset \mathbb{R}$
$m^*(A) = m^*(A_+) + m^*(A_-)$
where $A_+ = A \cap (0, \infty)$, $A_- = A \cap (-\infty, 0]$

(1) first $\because A = A_+ \sqcup A_-$ $\therefore$ by subadditivity,
$m^*(A) \le m^*(A_+) + m^*(A_-)$

(2) To show $m^*(A) \ge m^*(A_+) + m^*(A_-)$, it suffice to show $\forall \varepsilon > 0$
$m^*(A) + \varepsilon \ge m^*(A_+) + m^*(A_-)$

non strict → more robust

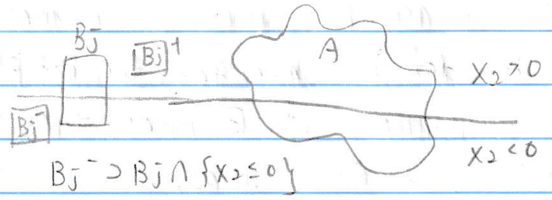
Consider an open cover of A by open boxes $\{B_j\}$ such that $\sum |B_j| \le m^*(A) + \varepsilon/2$

Define $B_j^+ = B_j \cap (0, \infty)$, $B_j^- = B_j \cap (-\infty, \varepsilon/2^{j+1})$.
then $B_j = B_j^- \cup B_j^+$, and $|B_j| + \varepsilon/2^{j+1} \ge |B_j^+| + |B_j^-| \ge |B_j|$

• $\bigcup B_j^+ \supset A_+$, $\bigcup B_j^- \supset A_-$

$m^*(A_+) + m^*(A_-) \le \sum |B_j^+| + \sum |B_j^-|$
$\le \sum_{j=1}^{\infty} (|B_j| + \varepsilon/2^{j+1})$
$\le (|B_j|) + \varepsilon/2 \le m^*(A) + \varepsilon/2 + \varepsilon/2 = m^*(A) + \varepsilon$

Try $n=2$ (to higher dimension)



$A_+ = A \cap \{x_2 > 0\}$
$A_- = A \cap \{x_2 \le 0\}$

need to show $m^*(A) + \varepsilon \ge m^*(A_+) + m^*(A_-)$

one can do the same ① get $\{B_j\}$ cover of A

with $m^*(A) + \varepsilon/2 \ge \sum |B_j|$

② $B_j^+ = B_j \cap \{x_2 > 0\}$

$B_j^- = B_j \cap \{x_2 < \varepsilon_j\}$

ε_j chosen s.t. $|B_j^-| + |B_j^+| \le |B_j| + \frac{\varepsilon}{2^{j+1}}$

There is a better, more systematic approach

Tao Ex 7.4.3 For $A =$ open box in $\mathbb{R}$

prove $m^*(A_+) + m^*(A_-)$

should be easy, since $m^*(A) = |A|$, $m^*(A_+) = |A_+| \dots$

by direct computation

For general A, for any $\varepsilon > 0$, find $\{B_j\}$

cover of A s.t. $m^*(A) + \varepsilon \ge \sum |B_j| = \sum_j |B_j^+| + |B_j^-|$

define $|B_j^+| = B_j \cap \{x_n > 0\}$, $B_j^- = B_j \cap \{x_n \le 0\}$

(may not be open)

$\because$ $A_+ \subset \bigcup B_j^+$ $\therefore$ $m^*(A_+) \le \sum m^*(B_j^+) = \sum |B_j^+|$

$A_- \subset \bigcup B_j^-$ $\therefore$ $m^*(A_-) \le \sum m^*(B_j^-) = \sum |B_j^-|$

$\therefore$ $m^*(A_+) + m^*(A_-) \le m^*(A) + \varepsilon$

Lemma 7.4.4. ; (property of measurable set) $\mathbb{R}^n \setminus E$

(a) if $E \subset \mathbb{R}^n$ measurable, then E^c is measurable.

(true by def, same test for E & E^c)

(b) translation invariance if E meas, then $\forall x \in \mathbb{R}^n$, $x+E$ is meas.

(pf: outer measure is translation invariant

$$\forall A \subset \mathbb{R}^n, \quad m^*(A) \overset{?}{=} m^*(A \cap (x+E)) + m^*(A \cap (x+E)^c)$$

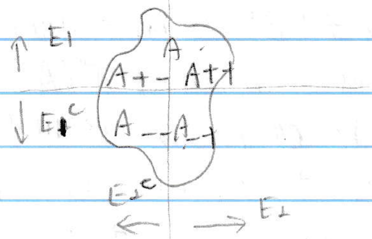
$$\Leftrightarrow \quad m^*(A-x) = m^*((A-x) \cap E) + m^*((A-x) \cap E^c)$$

this holds.

(c) If E_1 and E_2 are measurable, then $E_1 \cap E_2$, $E_1 \cup E_2$ are measurable

(pf WTS: $\forall A \subset \mathbb{R}^n$.

$$m^*(A) = m^*(A \cap (E_1 \cap E_2)) + m^*(A \setminus (E_1 \cap E_2))$$



$A_{++} = A \cap E_1 \cap E_2$. $\quad A_{+-} = A \cap E_1 \cap E_2^c$

$A_{--} = A \cap E_1^c \cap E_2^c \quad A_{-+} = A \cap E_1^c \cap E_2$

$$A = A_{++} \sqcup A_{+-} \cup A_{-+} \sqcup A_{--}$$

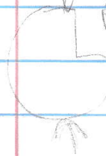
$$(*) \Leftrightarrow \quad m^*(A) = m^*(A_{++}) + m^*(A_{+-} \cup A_{-+} \cup A_{--})$$

we can show

$$m^*(A) = m^*(\underbrace{A_{+-} \cup A_{++}}_{A \cap E_1}) + m^*(\underbrace{A_{-+} \cup A_{--}}_{A \cap E_1^c})$$

using E_1 is measurable

break it into four pieces



$$m^*(A_{+-} \cup A_{++}) = m^*(A_{+-}) + m^*(A_{++}) \quad \because E_2 \text{ measurable}$$

$$m^*(A_{-+} \cup A_{--}) = m^*(A_{--}) + m^*(A_{-+})$$

(still need some more argument

$$m^*(A_{+-} \cup A_{-+} \cup A_{--}) = m^*(A_{+-}) + m^*(A_{-+}) + m^*(A_{--})$$

$A' =$

could become sum of four pieces

$A'_{--} = A_{--}$

A'_{++} is empty

(d) (Boolean algebra)

finite intersection / union preserve measurability

(using induction on number of operand)

(e) every box (open or closed) is measurable.

$([a,b] = [a,\infty) \cap (-\infty, b].)$

$a + [0,\infty)$ $b + (-\infty, 0]$

boxes are intersections of (translated) half spaces.

(f) if $m^*(E) = 0$, then E is measurable

(pf. $\forall A \subset \mathbb{R}^n$ $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$

only need to show $m^*(A) \geq m^*(A \cap E)$
$+ m^*(A \setminus E)$

$\because m^*(A \cap E) \leq m^*(E) = 0$ $\therefore m^*(A \cap E) = 0$

$(*) \Leftrightarrow m^*(A) \geq m^*(A \setminus E)$ which is true by monotonicity

- Lemma 7.4.5 (finite additivity) if $E_1, \ldots, E_n$ are disjoint measurable set, then $\forall A \subset \mathbb{R}^n$ $E = \bigcup_{i=1}^{n} E_i$

$$m^*(A \cap E) = \sum_{i=1}^{n} m^*(A \cap E_i)$$