Lec 4

(about lebesgue measure)

- one remark about last homework : we need

$m^*(\text{any box}) = \text{volume of a box}$

we know $m^*(\text{closed box}) = \text{volume}$

$m^*(\text{open box}) = \text{volume}$

If we have any "half open half closed" box, B.

$B^\circ \subset B \subset \bar{B}$, then.

$vol(B^\circ) = m^*(B^\circ) \le m^*(B) \le m^*(\bar{B}) = vol(\bar{B})$

$\therefore m^*(B) = vol(\bar{B}) = vol(B^\circ) = \prod (b_i - a_i)$

arbitrary measure : need to use axioms instead to prove.

- Lemma 7.4.7. If A, B are measurable, $A \subset B$.

then $B \setminus A$ is measurable, and

$m^*(B \setminus A) = m^*(B) - m^*(A)$

pf $B \setminus A = B \cap A^c$

($\because A$ is measurable $\therefore A^c$ is measurable)

($\because B$ and A^c is measurable $\therefore B \cap A^c$ is measurable.)

- WTS $m^*(B) = m^*(A) + m^*(B \setminus A)$

this follows from measurability of A

applied to test set B.

$\left(\because m^*(B) = m^*(\underbrace{B \cap A}_{= A}) + m^*(B \cap A^c)\right)$

- Prop (Countable Additivity) Let $\{E_j\}_{j=1}^{\infty}$ be a countable collection of disjoint measurable set

WTS

- $E = \bigcup_{j=1}^{\infty} E_j$ is measurable
- $m^*(E) = \sum_{j=1}^{\infty} m^*(E_j)$

pf :

• pf: To pv measurability, we want to show,

$(*)$ $\forall A \subset \mathbb{R}^n$, $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$

• Define $F_N = \bigsqcup_{j=1}^{N} E_j$, we know

  • F_N is measurable, (finite union of countable set)

  • $m^*(F_N) = \Sigma\, m^*(E_j)$

○ If we replace E by F_N, $\because E \supset F_N$, $E^c \subset F_N^c$

$\therefore$ $m^*(A \cap E) \geq m^*(A \cap F_N)$ (need fixing)

$m^*(A \cap E^c) \leq m^*(A \cap F_N^c)$

• To prove $(*)$ we need "$\underset{\downarrow\ \text{subadditivity}}{\leq}$" easy and "$\geq$"

$$m^*(A \cap E) \leq \Sigma_{j=1}^{\infty} m^*(A \cap E_j) \quad \text{by countable subadditivity}$$
$$= \sup_N \left(\Sigma_{j=1}^{N} m^*(A \cap E_j) \right)$$
$$= \sup_N m^*(A \cap F_N) \quad \text{by finite additivity}$$

Thus: $m^*(A \cap E) + m^*(A \cap E^c)$

$$\leq [\sup_N m^*(A \cap F_N)] + m^*(A \cap E^c)$$
$$\leq \sup_N (m^*(A \cap F_N) + m^*(A \cap E^c))$$
$$\leq \sup_N (m^*(A \cap F_N + m^*(A \cap F_N^c))$$
$$= \sup_N [m^*(A)] = m^*(A)$$

approx → name X → sup.

○ $m^*(E) \leq \Sigma_j m^*(E_j)$ by subadditivity

$m^*(E) \underset{\uparrow\ \text{by monotonicity}}{\geq} m^*(F_N) = \Sigma_{j=1}^{N} m^*(E_j)$

sup over N: we have $m^*(E) \geq \Sigma_{j=1}^{\infty} m^*(E_j)$

$\therefore m^*(E) = \Sigma_{j=1}^{\infty} m^*(E_j)$

- Prop 7.4.9. The set of measurable sets forms a σ-algebra, i.e. given any collection Ω_j of measurable sets, $\bigcap_{j=1}^{\infty} \Omega_j$ and $\bigcup_{j=1}^{\infty} \Omega_j$ are measurable

pf. Let's consider $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. Let $\Omega_N = \bigcup_{j=1}^{N} \Omega_j$ let $E_N = \Omega_N \setminus \Omega_{N-1}$

then $\{\Omega_N\}$ are measurable, $\{E_N\}$ are measurable.

Since $\Omega = \bigcup_{j=1}^{\infty} E_j$ $\therefore$ Ω is measurable.

- $\bigcap_{j=1}^{\infty} \Omega_j = \left(\bigcup_{j=1}^{\infty} \Omega_j^c \right)^c$ $\because$ complement & countable union preserves measurability

$\therefore$ this is measurable

- Lemma 7.4.10: All open sets in $\mathbb{R}^n$ can be written as a countable union of open boxes.

[Recall some topology:

- topology for a metric space (X,d)
  - open ball $B(x,r) = \{ y \in X \mid d(y,x) < r \}$
    $x \in X$, $r > 0$ real
- open sets in X are generated from open balls by taking finite intersections and arbitrary unions.

- equivalently, $U \subset X$ is open iff $\forall x \in U$ $\exists r$ s.t. $B(x,r) \subset U$

- topology on product space:
  If X, Y are topological spaces, then $X \times Y$ endow with product topology, i.e.
  $W \subset X \times Y$ is open if $\forall (x,y) \in W$ $\exists U \subset X$, $V \subset Y$ open s.t. $(x,y) \in U \times V \subset W$

- topology on $\mathbb{R}^2$
  - can be generated by open balls (using Euclidean metric on $\mathbb{R}^n$)
  - can be generated by open boxes.

pf: Consider the set of "rational boxes". A box $\prod_{i=1}^{n}(a_i, b_i)$ is rational if $a_1, b_1, \ldots, a_n, b_n \in \mathbb{Q}$

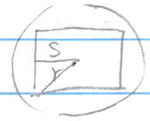
? • {The collection of rational boxes} $\subset \mathbb{Q}^{2n}$ is countable.

($\because$ $\mathbb{Q}$ is countable, finite product of countable set is countable and subset of countable set is countable)

- suffices to show that every open set in $\mathbb{R}^n$ is a union of rational boxes.
  i.e. If U is open, $x \in U$, we want to find a rational box B, s.t $x \in B \subset U$

$\because$ U is open $\therefore \exists r > 0$ s.t. $x \in B(x, r) \subset U$

claim: $\exists$ rational box B s.t $x \in B \subset B(x, r)$



Prop: Open sets in $\mathbb{R}^n$ are measurable

- open boxes are measurable. (intersection of halfspaces)
- a open set is a countable union of open boxes.

Discussion

Alternative definition of measurable sets

Def 2: A subset $E \subseteq \mathbb{R}^n$ is measurable if $\forall \varepsilon > 0$, $\exists$ an open set U s.t.

open cover. $m^*(U \setminus E) < \varepsilon$

in discussion, pv that all properties of measurable sets (Lemma in 7.4) can be derived using this definition

$U \setminus E$ $U \cap E$

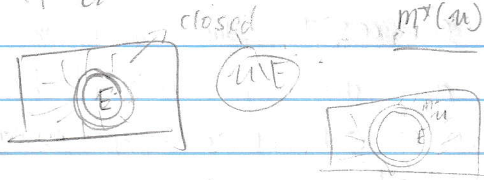
(a) $\forall \varepsilon > 0$ $\exists U$ s.t. $m^*(U \setminus (\mathbb{R}^n \setminus E)) < \varepsilon$

$= m^*(U \cap E)$ ϕ

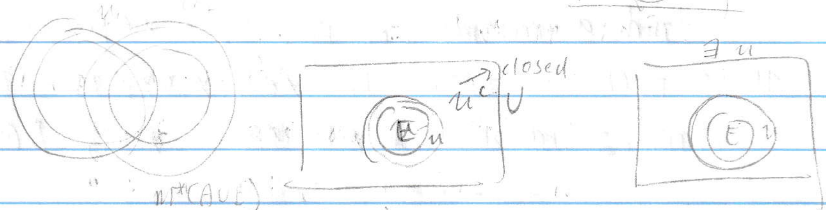
let $\varepsilon > 0$

closed $U \setminus E$

$m^*(U) - m^*(U \setminus E) \leq \varepsilon$



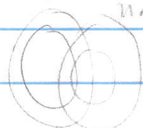
$U^c \to$ closed



$m^*(A \cup B) = m^*(U \cup V)$

$A \cup B$

$m(A) + m(B)$



let $\varepsilon > 0$

WTS $\exists$ $m^*(U \setminus E_1 \cap E_2) < \varepsilon$

$\exists U$ s.t. $m^*(U_1 \setminus E_1) < \varepsilon/2$

$m^*(U_2 \setminus E_2) < \varepsilon/2$

$m^*(U \setminus (E_1 \cap E_2)) \leq m^*(U \setminus E_1 \cup U \setminus E_2) \leq \varepsilon/2 + \varepsilon/2$