Lec.

Abstract Measure Theory $\neq$ Map$(S, \{0,1\})$

- S set 2^S be the set of all subsets
- (σ-algebra) Definition: A subset $M_S \subset 2^S$ is called a σ-algebra if ① $\phi \in M_S$.
  ② If $A_1, A_2, \dots \in M_S$ countable collection then
  ③ $A \in M_S$, $A^c \in M_S$, $\bigcup_{n=1}^{\infty} A_n \in M_S$

  (S, M_S) is called a "measurable space"
- Def: A measure w on (S, M_S) is a function
  $w: M_S \to [0, +\infty]$
  s.t. ① $w(\phi) = 0$
  ② countable additive if $A_1, A_2, \dots \in M_S$ disjoint
  then $w(\bigcup A_n) = \sum w(A_n)$

  (S, M_S, w) is called a measurable space.

Measurable function: (stronger condition than taw construction)

Given (X, M_X) and (Y, M_Y), we say
a measurable map/function from X to Y, is
$f: X \to Y$ such that for all $E \in M_Y$.
we have $f^{-1}(E) \in X$ is measurable.

- If $(X, M_X) \xrightarrow{f} (Y, M_Y) \xrightarrow{g} (Z, M_Z)$

  $g \circ f$ is measurable.

If S is a topological space, i.e. $T_S \subset 2^S$
collection of open subsets
then, there exists a minimum σ-algebra containing T_S
$\langle T_S \rangle$ Borel σ-algebra.

Q: on $\mathbb{R}^n$, is borel measurable equivalent to lebesgue measurable.

if borel meas. then lebesgue meas.

differs in a null set.

• Def. if S is a Topological space.
if $U_1, U_2, \ldots$ is a countable collection of
• open sets, then $\bigcap_{n=1}^{\infty} U_n$ is called a G_δ-set.

• If $F_1, F_2, \ldots$, are closed sets, the $\bigcup_{n=1}^{\infty} F_n$ is called a F_σ-set.

• Let S be a set. an outer measure ω on S is a function $\omega: 2^S \to [0,\infty]$

① $\omega(\phi) = 0$
② if $A \subset B$, then $\omega(A) \le \omega(B)$
③ Countable subadditivity, if $A_1, A_2 \ldots$
$\omega(\bigcup_n A_n) = \sum \omega(A_n)$

• construction: given ω.
def: $\mathcal{M}_S \subset 2^S$. $E \in \mathcal{M}_S$ if $\forall X \subset S$. $\omega(X) = \omega(X \cap E) + \omega(X \cap E^c)$

• Thm 5 (pugh)
• $\mathcal{M}_S$ is a σ-algebra.
• ω on $\mathcal{M}_S$ satisfies countable additivity
$(S, \mathcal{M}_S, \omega)$ is a measure space.

• a subset $E \subset S$, with $\omega(E) = 0$, is called the "zero set" "null set"

• Lemma: ② $\forall A \subset S$, $\omega(A \cup E) = \omega(A)$
① If $E' \subset E$ then E' is null. (monotonicity)

② $\omega(A \cup E) \le \omega(A) + \omega((A \cup E) \cap A^c)$
$= \omega(A) + \omega(E \cap A^c) = \omega(A)$
③ $\forall A \subset S$. , $\omega(A \cap E^c) = \omega(A)$
④ $\omega(E) = 0 \Rightarrow E$ is measurable.
pf: $\forall A \subset S$ $\omega(A) = \omega(A \cap E) + \omega(A \cap E^c)$
$= 0 + \omega(A)$

⑤ If Z is a null set, then F is measurable iff $F \cup Z$ is measurable

pf: WTS $\forall A \subset S$ $\quad w(A) = w(A \cap (F \cup Z)) + w(A \cap (F\cup Z)^c)$

$$= w((A\cap F) \cup (A \cap Z)) + w(A \cap F^c \cap Z^c)$$

$$= w(A \cap F) + w(A \cap F^c)$$

$$= w(A)$$

Slogan: adding / subtracting null set does not affect measurability.

- $\{0\} \times \mathbb{R} \subset \mathbb{R}^2$ has measure 0

pf: $\forall \varepsilon > 0$ $\exists$ open box cover of $\{0\}\times\mathbb{R}$ such that $\sum |B_j| < \varepsilon$

$$B_n = \left(\frac{-\varepsilon}{2^{2n+2}}, \frac{\varepsilon}{2^{2n+2}}\right) \times (-2^n, 2^n)$$

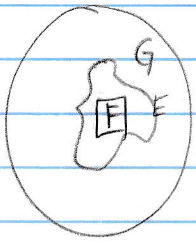
$|B_n| = \varepsilon/2^n$.

($\mathbb{R}^n$ lebesgue measure)

- Thm. $\forall E \subset \mathbb{R}^n$.

E is lebesgue measurable.

$\Longleftrightarrow$ $\exists$ F_σ-set F and G_δ-set G s.t. $F \subset E \subset G$, and $m(G \setminus F) = 0$



pf: assume E is bounded.

$\Rightarrow$ assume E is lebesgue measurable, and $E \subset \mathbb{R}$ big box.

$\forall n$, $\exists$ box covering $\{B_j^{(n)}\}$ of E, s.t. $\sum |B_j^{(n)}| < m(E) + 1/n$

let $\bigcup_j B_j^{(n)}$

similarly, let $V_n \supset \mathbb{R}\setminus E$, s.t. V_n open, $V_n \subset \mathbb{R}$

and $m(\mathbb{R}\setminus E) \le m(V_n) \le m(\mathbb{R}\setminus E) + 1/n$

$m(E) \le m(U_n) \le m(E) + 1/n$.

$F_n = \mathbb{R} \setminus V_n$ $\leftarrow$ closed $\quad F_n \subset E \quad F = \bigcup F_n$

$m(G\setminus F)$ $\quad m(F_n) \le m(E) \le m(F_n) + 1/n$.

$\le$

$\Rightarrow m(U_n \setminus F_n) \le 2/n$