- Lebesgue integral:

Recall: • Def: $f: \mathbb{R} \to +(0,\infty)$ is measurable if

$\mathcal{U}f \subset \mathbb{R}^2$ is measurable $\quad \mathcal{U}f = \{(x,y) \mid 0 \le y < f(x)\}$

? $\Updownarrow$

f^{-1}(open set) is measurable. (Tao)

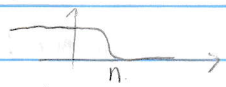
Ans: section 7

f is integrable if $m(\mathcal{U}f) < \infty$

- (upward) monotone convergence thm: if $f_n \nearrow f$ a.e. then $\int f_n \nearrow \int f$
- (downward) " " " : if $f_n \searrow f$ a.e. if f_1 is integrable then $\int f_n \searrow \int f$.

why?

contradiction



$$f_n(x) = \begin{cases} 1 & x \le n \\ 0 & x > n. \end{cases}$$

$\int f_n \to \infty \quad \int f = 0$

Dominated convergence thm:

- suppose $f_n \to f$ a.e.

suppose $g \ge f_n \ \forall n \quad \int g < \infty$

then $\int f_n \to \int f$

pf: $\bar{f}_n(x) = \sup\{f_k(x) \mid k \ge n\}$

$\underline{f}_n(x) = \inf\{f_k(x) \mid k \ge n\}$

$\underline{f}_n(x) \le f_n(x) \le \bar{f}_n(x)$

$\mathcal{U}\bar{f}_n = \bigcup_{m \ge n} \mathcal{U}f_m$

$\forall x \quad \lim f_n(x) = \lim\inf f_n(x) = \lim \underline{f}_n(x) = \lim\sup f_n(x) = \lim \bar{f}_n(x)$

① • $\underline{f}_n \nearrow f$ $\Downarrow$ $\int \underline{f}_n \nearrow \int f$

② • $\bar{f}_n \searrow f$ + $\int \bar{f}_n < \infty$ $\because$ $\forall x \quad [0, \sup_{m \ge n} f_m(x)) = \bigcup_{m \ge n} [0, f_m(x)) = \bigcap_{m \ge n} \hat{\mathcal{U}} f_m$. $\quad \int \bar{f}_n = \int f$

$\forall x \quad [0, \inf f_m] = \bigcap_{m \ge n} [0, f_m(x)]$

$\because g$ is integrable $\therefore \int g < \infty$

$g \ge f_n \ \forall n \ \therefore \ g \ge \bar{f}_n$

Q: if $f_n \to f$ a.e.

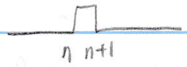
$\int f_n < M$

what can we say?

claim $\int f < M$.

f_n: integrable ($\because$ $\underline{f_n} \leq f_n$ $\int \underline{f_n} \leq \int f_n < \infty$

$\underline{f_n}, \overline{f_n}$: measurable

f_n: 

$\underline{f_n}(x) = 0$ $\int \underline{f_n} = \int f$ $\int \overline{f_n} \neq \int f$.

$\overline{f_n}(x) =$ 

Fatou's Lemma:

- $f_n: \mathbb{R} \to [0, \infty)$

  f_n: measurable

Then $\int \liminf f_n \leq \liminf \int f_n$.

By definition $\liminf f_n = \lim \underline{f_n} = f$

we know by monotone convergence

$\int \lim \underline{f_n} = \lim \int \underline{f_n}$

$\underline{f_n} \leq f_m \ \forall m > n$ $\int \underline{f_n} \leq \int f_m$, $\forall m \geq n$. $\Rightarrow \int \underline{f_n} \leq \inf_{m \geq n} \int f_m$

TB completed.

- Thm: if f, g are measurable: $\mathbb{R} \to [0, \infty)$

  then $\int f + g = \int f + \int g$

- Recall: mesomorphism
  - measurability - preserving maps

    $f: \mathbb{R}^n \to \mathbb{R}^n$ (preserves measurability and measure)
  - mestometry: measure - preserving bijection

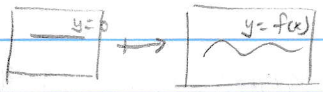
    $f: \mathbb{R}^n \to \mathbb{R}^n$. if $E \subset \mathbb{R}^n$ measurable then.

    $m(E) = m(f(E))$

if $f: \mathbb{R} \to \mathbb{R}$ we define

$T_f : \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (x, y+f(x))$

T_f :

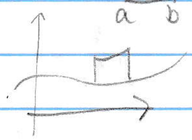


- note: T_f has inverse.

Thm (3T) T_f is isometry $f: \mathbb{R} \to [0,\infty)$ measurable.

- Let $R = h\{[a,b]\} \subset \mathbb{R}^2$

$T_f R =$

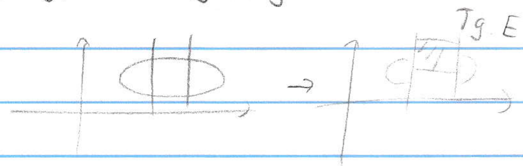


$T_f R = \mathcal{U}(f + h\chi_{[a,b]}) - \mathcal{U}(f)$

suffice to show $g = h\chi_{[a,b]}$ is measurable

$g = h \cdot \chi_{[a,b]}$

$\mathcal{U}(f+g) = T_f \hat{\mathcal{U}_g} = T_g \hat{\mathcal{U}_f}$



claim T_g preserves measurability

(measurable set translation invariant)

assume $f = f \cdot \chi_{[a,b]}$

claim $\sqcup T_f R = T_g(\mathcal{U}f) \sqcup R$

$\therefore m(\mathcal{U}f) + m(T_f R) = m(T_g \mathcal{U}f) + m(R)$

$= m(\mathcal{U}f) + m(R)$

T_g is mesometry.

- claim : T_f never increase the outer measure

$\forall A \subseteq \mathbb{R}^2 \ \forall \varepsilon > 0 \ \exists$ boxes $\{R_i\}$ s.t. $\sum m(R_i) \le m^*(A) + \varepsilon$

$\bigcup R_i \supset A$

$\therefore m^*(T_f A) \le \sum_{i=1}^{\infty} m(T_f R_i) = \sum_{i=1}^{\infty} m(R_i) \le m^*(A) + \varepsilon$

$\therefore m^*(T_f A) \le m^*(A)$ $\quad \because \psi$ is mesometry

$T_{-f} = \psi \circ T_f \circ \psi$, $\psi: (x,y) \mapsto (x,-y)$ $\quad \because T_{-f}$ preserves measurability

$m^*(T_{-f}(A)) \le m^*(A)$

$m^*(A) = m^*(T_f T_{-f} A) \leq m^*(T_{-f} A) \leq m^*(A)$

$\therefore m^*(T_f A) \leq m^*(A)$

$m(\mathcal{U}(f+g)) = m(T_f \mathcal{U}_g \sqcup \mathcal{U}f)$

$= m(T_f \mathcal{U}_g) + m(\mathcal{U}f)$

$= m(\mathcal{U}_g) + m(\mathcal{U}f)$ $\quad \int f+g = \int f + \int g$

Cor. if $\{f_n\} : \mathbb{R} \to [0, \infty)$ is a seq. of integrable fun

then $\int \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int f_k$.

pf: Let $F_k = \sum_{i=1}^{k} f_i$ $\quad F_k \nearrow F$ $\quad \int F_k \nearrow \int F$

$\int F_k = \sum_{i=1}^{k} \int f_k$ thus $\lim_{k\to\infty} \int F_k = \sum_{i=1}^{\infty} \int f_i$