Math105 Lec Feb, 17

1. if f, g are simple functions

$$\int f+g = \int f + \int g$$

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recall

$f = \sum c_i 1_{E_i}(x)$ if f is a simple function

$\int f = \sum c_i m(E_i)$ E_i meas.

$E_i \cap E_j = \phi$ $c_i > 0$

$$f = \sum_{i=1}^{N} c_i 1_{E_i}(x) \quad c_i > 0$$

$$g = \sum_{i=1}^{M} d_i 1_{E_j}(x) \quad d_i > 0$$

define $E_0 = \mathbb{R}^n \setminus E_1 \cup \cdots \cup E_N$ $c_0 = 0$

$F_0 = \mathbb{R}^n \setminus F_1 \cup \cdots F_M$ $d_0 = 0$

then $\mathbb{R}^N = \bigcup_{i=0}^{N} \bigcup_{j=0}^{M} E_i \cap F_j$ $\quad f+g = \sum_{i,j=0}^{N,M} (c_i + d_j) 1_{E_i \cap F_j}$

$$\rightarrow \int f+g = \sum_{i=0}^{N} \sum_{j=0}^{M} (c_i + d_j) m(E_i \cap F_j)$$

$$= \sum_{i,j} c_i m(E_i \cap F_j) + \sum_{i,j} d_j m(E_i \cap F_j)$$

$$= \sum_i c_i m(E_i) + \sum_j d_j m(F_j)$$

$$= \int f + \int g$$

Tao 8.2 integration of nonmeasurable fun.

Def. Let $f \geq 0$ be measurable $f: \Omega \to [0, \infty)$

$$\int f = \sup \{ \int s \mid s \text{ simple } s \geq 0, \ s \leq f \}$$

Q: just from def. is it true that a seq. of simple function s_n $0 \leq s_1 \leq s_2 \leq \cdots$

$$\sup_{n \geq 1} (s_n(x)) = f(x)$$

A: true (as we see later), but not obvious from definition

<u>Prop. 8.2.6</u> if $f, g : \Omega \to [0, \infty]$

(1) $\int f \geq 0$, and $\int f = 0$ iff $f(x) = 0$ almost everywhere.

a.e. $\left(\begin{array}{l} \text{i.e. } \exists Z \subset \Omega \text{ null set} \\ \text{s.t. } f|_{\Omega \setminus Z} = 0 \end{array} \right)$

(2) $\forall c > 0$ $\int c \cdot f = c \int f$

pf

(1) if $f(x) \geq 0$, $f(x) = 0$ a.e. then any simple function

$$s = c_i \mathbb{1}_{E_i}(x) \quad c_i > 0$$

$$\underset{\substack{\uparrow \\ \text{null}}}{E_i \subset Z} = \{ x \mid f(x) > 0 \}$$

$$\therefore \int s = \sum c_i m(E_i) = c_i \cdot 0 = 0$$

(c) $f \leq g \Rightarrow \int f \leq \int g$

(d) if $f = g$ a.e. then $\int f \leq \int g$

$$\bar{f} = \max(f, g)$$

$\underline{f} = \min(f, g)$

$\int \bar{f} \geq \int f \geq \int \underline{f}$

$\int \bar{f} \geq \int g \geq \int \underline{f}$

Pf: Let $Z = \{x \mid f(x) \neq g(x)\}$

$Z^c = \Omega \setminus Z$

$\forall s$ simple $\quad \int s = \int s 1_{Z^c} = \int f$

$= \int f \cdot 1_{Z^c}$

Can delete measure zero set without affecting integration $\quad = \int g 1_{Z^c} = \int g$

Thm. Given measurable functions

$f : \Omega \to [0, \infty]$ $\qquad 0 \leq f_1(x) \leq f_2(x) \leq \cdots$

$f_n : \Omega \to [0, \infty]$

then ① $\int f_1(x) \leq \int f_2(x) \leq \cdots$

② $\int \sup f_n(x) = \sup \int f_n$

$\because \sup f_n(x) \geq f_n(x) \quad \forall n.$ (with $\sup f_n(x) = f(x)$)

$\therefore \int f \geq \int f_n \ \forall n \quad \therefore \int f \geq \sup \int f_n$

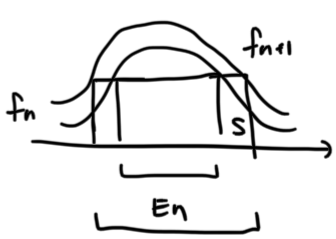
want to show $\quad \int f \leq \sup_n \int f_n$

$\Leftrightarrow \forall s$ simple $\quad 0 \leq s \leq f$

$\int s \leq \sup_n \int f_n$

$\Leftrightarrow \forall \ 1 > \varepsilon > 0 \quad \forall s$ simple

$(1-\varepsilon) \int s \leq \sup_n \int f_n$



define $\quad s(x) \leq \sup f_n = f$

$E_n = \{x \in \Omega \mid f_n(x) \geq (1-\varepsilon) s(x)\}$

then $\subset E_n \subset E_{n+1} \subset \cdots$ and $\bigcup E_n \subset \Omega$

E_{n+1}

$$\int_{E_n} (1-\varepsilon) s_n \le \int_{E_n} f_n \le \int_\Omega f_n \le \sup_n \int_\Omega f_n$$

Q : $\lim_{n\to\infty} \int_{E_n} (1-\varepsilon) s = \int_\Omega (1-\varepsilon) s$?

A : Yes.

$$\int_{E_n} s = \sum c_i m(F_i \cap E_n)$$

$\because E_n \nearrow \Omega$

$\therefore E_n \cap F \nearrow \Omega \cap F = F$

$$\lim \int_{E_n} s = \sum c_i m(F_i) = \int_\Omega s$$

$\therefore m(E_n \cap F) \nearrow m(F)$

$$\therefore (1-\varepsilon) \int_\Omega s_n = \int_\Omega (1-\varepsilon) s_n \le \sup_n \int_\Omega f_n$$

<u>prop.</u> if $f, g : \Omega \to [0,\infty]$ measurable

then $\int f+g = \int f + \int g$

<u>pf:</u> let f_n be sequence of simple func. $s_n \nearrow f$

t_n .... $t_n \nearrow g$

then by MCT $\lim \int s_n = \int f$, $\lim \int t_n = g$

$$s_n + t_n \nearrow f+g$$

$$\lim \int s_n + t_n = \int f+g$$

$$\because \int s_n + t_n = \int s_n + \int t_n$$

$$\therefore \lim \int t_n + s_n = \lim \left(\int s_n\right) + \lim \left(\int f t_n\right)$$

$$= f+g$$

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if $g_1, g_2, \cdots$ are nonegative meas. functions

then $$\int_\Omega \sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} \int g_n(x)$$

pf: define $f_n = \sum_{i=1}^{n} g_m(x)$

then $$\int \sup f_n = \sup \int f_n = \sup_N \int \sum_{n=1}^{N} g_n$$
$$= \sup_N \sum_{n=1}^{N} \int g_n$$

If $f : \Omega \to [0, \infty]$ meas.

$\int f < \infty$

then $f(x)$ is finite a.e.

$\ulcorner$ $\exists Z$ null set s.t. $f|_{\Omega \setminus Z}$ is finite

i.e. $f^{-1}(\infty)$ is a null set

Corr. (Borel-Cantelli)

if $\Omega_1, \Omega_2, \ldots$ are meas. s.t. $\sum m(\Omega_i) < \infty$

then $\{x \mid x$ belongs to infinitely many $\Omega_n\}$ is

a null set

$$m(\Omega_i) = \int_{\Omega_i} 1 = \int_\Omega 1_{\Omega_i}$$

$$\sum_{i}^{\infty} m(\Omega_i) = \sum_{i}^{\infty} \int_{\Omega_i} 1_{\Omega_i} = \int \sum_{i=1}^{\infty} 1_{\Omega_i}(x) < \infty$$