Lec.

Recall: ∘ Lebesgue integration over $\mathbb{R}^n$.

for nonnegative measurable $f: \mathbb{R} \to \mathbb{R}$

Today: ∘ extend to $f: \mathbb{R}^n \to \mathbb{R}$

∘ Comparison between Riemann integral & Lebesgue integral

∘ Fubini Thm.

∘ Given $f: \mathbb{R}^n \to \mathbb{R}$ measurable.

define $E_+ = \{x \in \mathbb{R}^n \mid f(x) > 0\}$ indicator fcn

$E_- = \{x \in \mathbb{R}^n \mid f(x) < 0\}$ $1_{E_+} = \chi_{E_+}$

$f_+ = f \cdot \chi_{E_+}$ $f_- = -f \cdot \chi_{E_-}$

then $f = f_+ - f_-$

· Def f is absolutely integrable if $\int |f| < \infty$

$\Leftrightarrow \int f_+ + \int f_- < \infty$

$\Leftrightarrow \int f_+ < \infty, \int f_- < \infty$

Thm: (dominated covergence thm)

Let $f_1, f_2, \dots f_n$ be a sequence of integrable fcn.

s.t. $f_n \to f$ ptwise, if $\exists F: \mathbb{R}^n \to \mathbb{R}$ non-negative integrable.

and $F \geq |f_n|$, then

$$\int \lim f_n = \lim \int f_n.$$

pf: recall: Fatou's Lemma.

if f_n is a seq. of non-neg. func.

$$\int \liminf f_n \leq \liminf \int f_n.$$

construct: $F + f_n \geq 0$, $F - f_n \geq 0$

Apply Fatou's Lemma to seq. $F + f_n$, $F - f_n$.

$$\int F + f = \int \liminf F + f_n \leq \liminf \left(\int F + \int f_n\right) = \int F + \liminf \int f_n.$$

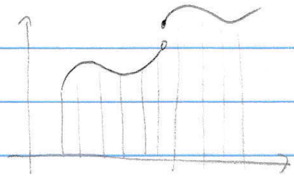
$$\int F - \int f = \int \liminf (F - f_n) \leq \liminf \left(\int F - \int f_n\right) = \int F - \limsup \int f_n.$$

$$\liminf \int f_n \geq \int f \geq \limsup \int f_n.$$

but $\liminf \int f_n \leq \limsup \int f_n$

hence $\int f = \lim \int f_n$.

∘ Riemann integral.



$f : [a,b] \to \mathbb{R}$

$(*)$

$\text{Riemann}\,\underline{\int} f = \sup_{\substack{P \text{ partition} \\ \text{over } [a,b]}} \left(\sum \inf f\big|_{[x_i, x_{i+1}]} \cdot |x_{i+1} - x_i| \right)$

$\{x_0 < x_1 < \cdots < x_n\}$, $x_0 = a$, $x_n = b$.

$\overline{\text{Riemann}\int} f = \underline{\text{Riemann}\int f}$ $\qquad$ $\text{Riemann}\,\overline{\int} f = \inf_{\substack{P \text{ partition} \\ \text{over } [a,b]}} \left(\sum \sup f\big|_{[x_i, x_{i+1}]} \cdot |x_{i+1} - x_i| \right)$

$1_{\mathbb{Q} \cap [a,b]}$ is not riemann integ.

Connection

∵ Riemann $\rightarrow$ simple function

$\text{Riemann} \int f \leq \int f$

fact: $\quad f \leq M 1_{[a,b]}$ bounded.

Let $P_1, P_2, \ldots,$ be a sequence of partition that realize the sup. in $(*)$ $\qquad P_{n+1}$ refines P_n.

Let $S_1, S_2, \ldots,$ be corresponding simple functions,

then $S_{n+1} \geq S_n$, Let $\underline{f} = \lim S_n$, by monotone convergence thm,

$\int \underline{f} = \lim \int S_n = \underline{\text{Riemann} \int f}$

consider the Rieman upper integral. $\overline{\text{Riemann}\int f}$

$\lim S_n =: \overline{f}$ (decreasing.)

$\int \overline{f} = \lim S_n = \overline{\text{Riemann}\int f}$

$\overline{\text{Riemann}\int f} = \int \overline{f} \geq \int f \geq \int \underline{f} = \text{Riemann}\int f$

Recall: ① If $\quad a_n \geq 0$ and $\sum a_n < \infty$,

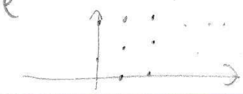
then any rearrangement of (a_n) is still summable.

let $M = \sum a_n$.

$\tilde{a}_n$ is an rearrangement the

$\tilde{S}_N = \sum_{n=1}^{N} \tilde{a}_n \leq S_{N'} = \sum_{n=1}^{N'} a_n < M$

sum. need not be finite



Recall: ② if $\sum_n \sum_m a_{n,m} < \infty$, $a_{n,m} > 0$

then $= \sum_{(n,m)} a_{n,m}$

③ if $g_n(x) \geq 0$ then

$$\sum_n \int_x g_n(x) = \int_x \sum_n g_n(x)$$

④ if f is a measurable function, $f: \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$

then "informal version" $\int \left[\int f(x,y)\,dy \right] dx$

$$= \int \left[\int f(x,y)\,dy \right] dx$$

but don't know if $\int f(x,y)\,dx$ is measurable

A: this is true

$$= \int_{\mathbb{R}^2} f(x,y)\,dx\,dy$$

sketch of proof

replace $\underline{f}$ by

$f_N = f \cdot 1_{[-N,N] \times [-N,N]}$

just need to prove for f_N with bounded support.

② replace f by

simple function with bounded support,

just need to prove this case.

③ replace simple func. by indicator function

$1_E(x,y)$

WTS $\int\int 1_E\,dx\,dy = \int 1_E\,dx\,dy = m(E)$

- if B is an open box $B = (a,b) \times (c,d)$

$$\int \left[\int 1_B(x,y)\,dx \right] dy$$

$$= (b-a)(c-d) = m(B)$$

$\forall \varepsilon > 0$ let

$\{B_n\}$ be a countable cover of E by open boxes.

s.t. $\sum_n m(B_n) < m(E) + \varepsilon$

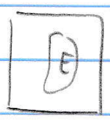
then $\int\left(\int 1_E \, dx\right) dy$ $\quad \because E \subset \bigcup B_n$ $\quad \therefore 1_E(x) \leq \sum 1_{B_n}(x)$

$$\leq \int\left[\int \sum_n 1_{B_n} \, dx\right] dy = \int \sum_n \int 1_{B_n}(x,y) \, dx \, dy$$

$$= \sum_n m(B_n) < m(E) + \varepsilon$$

Thus for $E \subset [-N,N] \times [-N,N] = R$

we have $\int\left(\int 1_E \, dx\right) dy \leq m(E)$


R

Let $E^c = R \setminus E$. Then.

$$\int\left(\int 1_{E^c} \, dx\right) dy \leq m(E^c)$$

$$1_{E^c} = 1_R - 1_E$$

$m(E) = m(R) - m(E) \Rightarrow m(R) - \int\left(\int 1_R dx\right) dy \leq m(R) - m(E)$

$\int\left(\int 1_R \, dy\right) dy = m(R)$

$\int\left(\int 1_E \, dx\right) dy = m(E)$

$= \int (1_E) \, dx \, dy$

Thm: (Fubini–Tonielli)

if $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ is absolutely integrable

i.e. $\int f(x,y) \, dx \, dy < \infty$

then $\int\left(\int f(x,y) \, dx\right) dy = \int f(x,y) \, dx \, dy$

$$= \int\left(\int f(x,y) \, dy\right) dx$$