Lec

Today : ∘ upper / lower lebesgue integral

for any $f: \mathbb{R}^n \to \mathbb{R}$ function (not necessarily meas.)

$\overline{\int} f = \inf\{ \int g,\ |g| \text{ integrable},\ g \geq f \}$

$\underline{\int} f = \sup\{ \int g,\ |g| \text{ integrable},\ g \leq f \}$

properties :

∘ $\overline{\int} f \geq \underline{\int} f$ ($\because g' \geq g''$ then $\int g' \geq \int g''$)

∘ if f is integrable $\int f = \overline{\int} f = \underline{\int} f$

Lemma : if $\overline{\int} f = \underline{\int} f$, then $\int f$ exist and equal to the common value

pf : Let $\overline{f_n}$ be integrable $\overline{f_n} \geq f$ s.t. $\overline{f_n} \leq A + 1/n$ $\overline{f_n} \searrow$

Let $\underline{f_n}$ " " $\underline{f_n} \leq f$ s.t. $\underline{f_n} \geq A - 1/n$. $\underline{f_n} \nearrow$

then define $F_+ = \lim_n \overline{f_n}$, $F_- = \lim_n \underline{f_n}$

$\int F_+ = \lim \int \overline{f_n} = A$ dominated convergence

$\int F_- = \lim \int \underline{f_n} = A$ (monotone applies to unsigned)

$F_+ \geq f \geq F_-$

$\because \int F_+ = \int F_-$ $\therefore \int F_+ - F_- = 0$

$F_+ = F_-$ almost everywhere

$\therefore f = F_+$ a.e. $\therefore f$ is integrable

Thm (Fubini thm) :

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be integrable.

Then $\exists$ $F(x)$ $G(y)$ integrable fcn.

s.t. for a.e. x, $F(x) = \int f(x,y)\,dy$

and for a.e. y $G(y) = \int f(x,y)\,dx$

and $\int f(x,y)\,dx\,dy = \int F(x)\,dx = \int G(y)\,dy$

Remark: slices of f may not be meas. thus $f(x,y) = 1_E(x,y)$ may not be a meas. func. when viewed as fnc of y fix x.

pf : ① only pv it for $F(x)$

② since $f(x,y) = \sup_N (\underbrace{f \cdot 1_{[-N,N] \times [-N,N]}}_{f_N})$

$\int f(x,y)\,dy$ // $\int \sup_N f_N(x,y)\,dy$

if statement is true for f_N i.e. $\exists F_N$ s.t. // DTC

then let $F = \sup F_N$, then $F(x) = \sup_N \int f_N(x,y)\,dy$

continue: replace f by f_N, just prove it for fcn with bounded support $\because f = f_+ - f_-$ where

$f_+ = f\chi_{\{f(x)>0\}}$

$f_- = -f\chi_{\{f(x)<0\}}$

reduce to non-negative.

③ simple fcn with bounded support in $[-N,N]^2$.

④ . .

pf: WTS statement is true for $f = 1_E$ where $E \subset [-N,N]^2$ meas.

claim: $\bar{\int}\left[\bar{\int} 1_E(x,y)\,dy\right]dx \le m(E)$

pf: need to show, for all $\varepsilon>0$, LHS $\le m(E)+\varepsilon$.

$\exists$ boxes $\{B_i\}$ s.t. $\sum |B_i| \le m(E)+\varepsilon$ and $\cup B_i$ covers E

note $f_1 \le f_2$, $\bar{\int} f_1 \le \bar{\int} f_2$

$\sum 1_{B_i} \ge 1_E \Rightarrow \bar{\int}\left[\bar{\int}(1_{B_i}(x,y))dy\right]dx$

$\ge \bar{\int}\left(\bar{\int} 1_E\,dy\right)dx$

$\sum m(B_i) \le m(E)+\varepsilon$

$E^c = [-N,N]^2 \setminus E$ $\qquad R = [-N,N]^c$

$\bar{\int}\left[\bar{\int} 1_{E^c}\,dy\right]dx \le m(E^c) = m(R) - m(E) = 4N^2 - m(E)$

$\|$

$\bar{\int}\left[\bar{\int}(1_R - 1_E)(x,y)\,dy\right]dx = 4N^2 - \underline{\int}\left(\underline{\int} 1_E(x,y)\,dy\right)dx$

$= \int 1_R\,dy - \int 1_E(x,y)\,dy$

$= 2N - \int 1_E(x,y)\,dy$

meas.

pf: WTS: statement is true for $f = 1_E$ where $E \subset [-N,N]^2$

claim $\bar{\int}\left[\bar{\int} 1_E(x,y)\,dy\right]dx \le m(E)$

$\underline{\int}\left[\underline{\int} 1_E(x,y)\,dy\right]dx \ge m(E)$

$\bar{F}(x) = \bar{\int} f(x,y)\,dy$

$\underline{F}(x) = \underline{\int} f(x,y)\,dy.$

$$\underline{\int}\underline{\int} 1_E \le \bar{\int}\left(\bar{\int} 1_E\,dy\right)dx \le m(E) \le \underline{\int}\left(\underline{\int} 1_E\,dy\right)dx$$

$\Rightarrow \quad \bar{\int}\underline{F} = \underline{\int}\underline{F} \Rightarrow \underline{F}$ is integrable $\int \underline{F} = m(E)$

similarly $\bar{\int}\bar{F} = \underline{\int}\bar{F} \Rightarrow \bar{F}$ is integrable. $\int \bar{F} = m(E)$

$\Rightarrow \quad \int \bar{F} = \int \underline{F} \Rightarrow \int(\bar{F} - \underline{F})\,dx = 0$

$\Rightarrow \bar{F}(x) = \underline{F}(x)$ for a.e. x.

thus for a.e. x. $\bar{\int} f(x,y)\,dy = \underline{\int} f(x,y)\,dy.$

thus $\int f(x,y)\,dy$ exist

like limsup, liminf

6.8 Pugh.

Vitali Covering Lemma.

Def. Let $A \subseteq \mathbb{R}^n$ be a subset. A covering $\mathcal{V} = \{V_\alpha\}_{\alpha \in I}$

may not be open.

is called vitali if $\forall p \in A$, $\forall r > 0$ $\exists V \in \mathcal{V}$ s.t. $\{p\} \subseteq V \subseteq B_r(p)$

Ex: $\mathbb{R}^n$, $\mathcal{V} = \{$all open balls with "$\mathbb{Q}$ center", $\mathbb{Q}$ radius$\}$

could do closed balls for Ex as well.

- Thm Let $A \subset \mathbb{R}^n$ be a bounded subset (measurable)

Let $\mathcal{V}$ be a Vitali cover of A cosist of closed balls closed balls.

Then $\forall \varepsilon > 0$ $\exists$ a countable subset of $\mathcal{V}$, $\{V_1, V_2, \dots\}$.

s.t. ① $V_i \cap V_j = \emptyset$ $\forall i \ne j$

② $\sum m(V_i) \le m(A) + \varepsilon$

③ $A \setminus \left(\bigcup_{i=1}^{\infty} V_i\right)$ is a nullset pf: ① pick open set $W \supset A$ s.t. $m(W) \le m(A) + \varepsilon$.

② Let $\mathcal{V}_1 = \{V \in \mathcal{V} : V \subset W\}$

$d_1 = \sup\{\operatorname{diam}(V), V \in \mathcal{V}_1\}$

pick $V_1 \in \mathcal{V}_1$ s.t. $\operatorname{diam}(V_1) > \frac{1}{2} d_1$

③ Let $W_2 = W \setminus \overline{V_1}$ (still open)

$\mathcal{V}_2 = \{V \in \mathcal{V}_1 : V \subset W_2\}$

$d_2 = \sup(\operatorname{diam}(V) \mid V \in \mathcal{V}_2)$

pick $V_2 \in \mathcal{V}_2$ such that $\operatorname{diam}(V_2) > \frac{1}{2} d_2$

.... get V_3, V_4 ....

claim: $A \setminus \bigcup_{i=1}^{\infty} V_i$ is a null set.

- Let $U_N = \bigcup_{i=1}^{N} V_i$, since $\sum m(V_i) = m(\bigcup_{i=1}^{\infty} V_i) \subset m(W) < \infty$

  $\therefore m(V_i) \to 0$

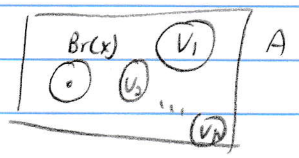
  $\therefore \operatorname{diam}(V_i) \to 0$

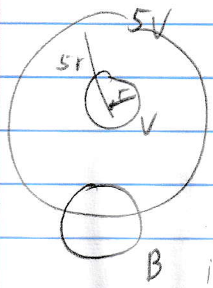
  $\Rightarrow d_i \to 0$

- claim2: $\forall N > 0$ $\bigcup_{k=N}^{\infty} 5 \cdot V_k \supset (A \setminus U_{N-1})$

pf of claim2, if not, then $\exists N$, $\exists x \in A \setminus U_{N-1}$ s.t.

$x \notin \bigcup_{k=N}^{\infty} 5 \cdot V_k$, $\exists B_r(x)$ s.t. $B_r(x) \cap U_{N-1} = \phi$





thus $B \not\subset \bigcup_{k=N}^{\infty} 5V_k$

pick $B \subset V_N$

s.t. $\{x\} \subseteq B \subset B_r(x)$

$\Rightarrow B \not\subset 5 \cdot V_N$

$\not\Rightarrow B \cap V_N = \phi$

B if $B \not\subset 5V$

and $B \cap V_N \neq \phi$

then $\operatorname{dian}(V_N) > \frac{1}{2} d_N$