Lec.

- Vitali Covering:

we say $\mathcal{V} = \{V_a\}$ is a Vitali Covering of $A \subset \mathbb{R}^n$, if $\forall p \in A$ $\forall r>0$ $\exists V \in \mathcal{V}$ s.t. $\{p\} \subseteq V \subset B_r(p)$

- V.C.L. If A is a bounded set, if $\mathcal{V}$ is a Vitali covering of A by closed balls. Then $\forall \varepsilon > 0$, we can have a countable collection $V_1, V_2, \cdots$ of $\mathcal{V}$

(1) $V_i \cap V_j = \phi$

(2) $\sum m(V_i) < m(A_i) + \varepsilon$

(3) $A \setminus (\cup V_i)$ is measure zero. how?

- pf. ① take W open s.t. $m(W) < m(E) + \varepsilon$.

② let $\mathcal{V}_1 = \{V \in \mathcal{V} : V \subset W\}$ $\Rightarrow$ still a vitali covering of A $\quad W_1 = W$; $\mathcal{V}_0 = \mathcal{V}$

$d_1 = \sup\{\operatorname{diam} V, V \in \mathcal{V}_1\}$

V_1 is any element in $\mathcal{V}_1$ s.t. $\operatorname{diam}(V_1) > d/2$

Let $W_2 = W_1 \setminus V_1$, $\mathcal{V}_2 = \{V \subset W_2, V \in \mathcal{V}_1\}$

$\mathcal{V}_n = \{V \subset W_n, V \in \mathcal{V}_{n-1}\}$

claim: $\bigcup_{k=N}^{\infty} 5 \cdot V_k \supset A \setminus U_{N-1}$

pf by contradiction

suppose false, then $\exists p \in A \setminus U_{N-1}$

but $p \notin \bigcup_{k=N}^{\infty} 5 V_k$ then $\exists B \in \mathcal{V}_N$ s.t. $p \in B$

Lec

other textbook: Folland (Rodon - Nikodyn derivative)

Desity theorem

- Recall

$E \subset \mathbb{R}^n$ meas.

$\delta(P, E) := \lim_{Q \downarrow P} \frac{m(E \cap Q)}{m(Q)}$ if it exists

density points of E:

all poinsts $p \in E$ s.t. $\delta(p, E) = 1$

Properties

1. $0 \le \delta \le 1$

2. $\delta(P, E) \exists \Rightarrow \forall \varepsilon > 0 \ \exists \ell > 0$ s.t. $\forall$ cube Q containing p with leghth $< \ell$.

$$\left| \frac{m(E \cap Q)}{m(Q)} - \delta(P, E) \right| < \varepsilon$$

Thm: If E meas. then almost all $p \in E$ are density points i.e. $\delta(p, E) = 1$ for almost all $p \in E$.

$$\underline{\delta}(P, E) = \liminf_{Q \downarrow p} \frac{m(Q \cap E)}{m(Q)}$$

Take $0 \le a < 1$

Let $E_a = \{p \in E \mid \underline{\delta}(p, E) < a\}$ strict equality ☆ necessary.

WTS $m^*(E_a) = 0$

pf: assume E bounded

$p \in E_a \Rightarrow \exists$ arbitiarily many small cubes Q containing p s.t. $\frac{m(Q \cap E)}{m(Q)} < a$ (by def of liminf)

$\forall r > 0$, we can find Q s.t. $p \in Q \subset B_r(p)$

S = set of all Q for all p

Then S is vitali coveringof E_a

By VCL, get $Q_1, Q_2, \cdots$

$dp(E) = \{p \in \mathbb{R}^n \mid \delta(p, E) = 1\}$

- Corr : $dp(E) \overset{0}{=} E$

  ($A \overset{0}{=} B$, i.e. $A \Delta B = (A \setminus B) \cup (B \setminus A)$ has measure 0)

pf : note

$dp(E) \cap E$ full measure in E

$dp(E^c) \cap E^c$ full measure in E^c

$dp(E) \cap dp(E^c) = \phi$

($\because$ $\forall Q$ cube $[Q : E] + [Q : E^c] = 1$)

thus $dp(E) \cap E^c \subset dp(E) \cap E^c \setminus dp(E^c) \subset E^c \setminus dp(E^c)$ null.

- Lemma if $E_1 \overset{0}{=} E_2$, then $dp(E_1) = dp(E_2)$

  pf: since $\forall Q$ cube, $[E_1 : Q] = [E_2 : Q]$

  thus p is density pt of E_1

  $\Leftrightarrow$ p is dp of E_2

  $E \overset{0}{=} dp(E)$ $\therefore dp(E) = dp(dp(E))$

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lebesgue Integrable

  (ie for any bounded closed set / compact set, $\int |f| < \infty$)

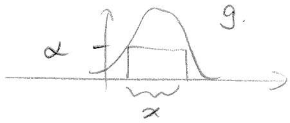
  $\forall p \in \mathbb{R}^n$, $\underline{\delta}(p, f) = \liminf \frac{1}{m(Q)} \int_Q f$ lower mean value

  $\overline{\delta}(p, f) = \limsup \frac{1}{m(Q)} \int_Q f$ upper " "

  if $\underline{\delta} = \overline{\delta}$, we have $\delta(p, f) = \underline{\delta}(p, f)$

Thm Let f be locally integrable ($f \in L_{loc}(\mathbb{R}^n)$)

then for a.e. $p \in \mathbb{R}^n$ $\delta(p, f)$ exists and $= f(p)$



Lemma: if $g : \mathbb{R}^n \to [0, \infty)$, integrable, $\forall \alpha > 0$

then $\alpha m^*(\{ p \in \mathbb{R}^n \mid \bar{\delta}(p, g) > \alpha \}) < \int g$.

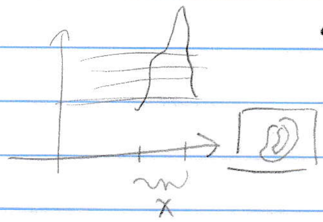
pf: $\forall p \in X(\alpha, g)$ $\exists$ arbitrary small cubes Q s.t. $\int g > \alpha$

such cubes form a Vitali covering,

Fix $\varepsilon > 0$, get a ε-efficient subcovering of X_α

$Q_1, Q_2, \dots$ $\quad \alpha \cdot m^*(X_\alpha) \le \alpha \cdot \sum_{i=1}^{\infty} m(Q_i) < \sum_i \int_{Q_i} g \le \int_{\cup Q_i} g \le \int g$

pf of Thm: • assume $f \ge 0$, assume f is supported in a closed unit cube X



• Fix an $\alpha > 0$, define X_k (for any $k \in \mathbb{Z}^+$)

$= f^{-1}([k\alpha, (k+1)\alpha])$

$X_{<k} = \bigcup_{j<k} X_j \qquad X_{>k} = \bigcup_{j>k} X_j$

Fix k,

for any cube Q

$$\frac{1}{mQ}\int_Q f = \frac{1}{m(Q)}\left(\int_{Q\cap X_{<k}} f + \int_{X_k} f + \int_{Q \cap X_{>k}} f\right)$$

$\forall\ p \in dp(X_k) \cap X_k$, we have

$$\limsup_{Q \downarrow p} \frac{1}{m(Q)} \int_{Q \cap X_{<k}} f \le k\alpha \, \overline{\lim_{Q\downarrow p}} \frac{m(Q \cap X_{<k})}{m(Q)} = 0 \quad (\because p \text{ is not a density pt of } X_k)$$

$$k\alpha \le \underline{\lim_{Q \downarrow p}} \frac{1}{m(Q)} \int_{Q \cap X_k} \le \overline{\lim_{Q\downarrow p}} \frac{1}{m(Q)} \int_{X_k \cap Q} f \le (k+1)\alpha.$$