HW10

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12.

Fix two real numbers a and b, $0 < a < b$. Define a mapping $\mathbf{f} = (f_1, f_2, f_3)$ of $\mathbf{R}^2$ into $\mathbf{R}^3$ by

$$\begin{aligned} f_1(s,t) &= (b + a\cos s)\cos t \\ f_2(s,t) &= (b + a\cos s)\sin t \\ f_3(s,t) &= a\sin s. \end{aligned}$$

(a)

Show that there are exactly 4 points $\mathbf{p} \in \mathbf{K}$ such that

$$(\nabla f_1)(\mathbf{f}^{-1}(\mathbf{p})) = \mathbf{0}$$

Proof. First note that

$$\nabla f_1 = \begin{pmatrix} -a\sin s\cos t \\ -(b + a\cos s)\sin t \end{pmatrix}$$

Now consider when $\nabla f_1 = (\mathbf{0})$, then either

$$\begin{cases} \sin t = 0 \text{ and } \sin s = 0 \\ \cos t = 0 \text{ and } \cos s = \frac{-b}{a} & \text{not possible since } b > a \end{cases}$$

thus $t, s \in \{\pi, 0\}$. □

(b)

Determine the set of all $\mathbf{q} \in \mathbf{K}$ such that

$$(\nabla f_3)(\mathbf{f}^{-1}(\mathbf{q})) = 0$$

Proof. We apply a similar approach to that of part (a), we first get

$$\nabla f_3 = a\cos s$$

Thus the corresponding solution goes is $s \in \{\pi/2, 3\pi/2\}$. □

(c)

Show that one of the points **p** found in part (a) corresponds to a local maximu of f_1, one corresponds to a local minimum and that the other two are neither. What about for (b)

Proof. We look at the second order total derivative and plug in the solutions obtained in (a)

$$D^2 f_1 = D \circ Df_1 = \begin{pmatrix} -a\cos s \cos t & a \sin s \sin t \\ a \sin s \sin t & (b + a\cos s)\cos t \end{pmatrix}$$

plugging in $t, s \in \{\pi, 0\}$. We get For (b) since we know the max value of f_3

t	s	description
π	π	saddle
π	0	min
0	π	saddle
0	0	max

would be when $\sin s = 1$, thus when $s = \pi/2$ we get the max, $s = 3\pi/2$ we get the min. □

13

Suppose $\mathbf{f}$ is a differentiable mapping of $\mathbf{R}^1$ into $\mathbf{R}^3$ such that $|\mathbf{f}(t)| = 1$ for every t. Prove that $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$.

Proof.

$$\begin{aligned} |\mathbf{f}(t)| &= 1 \\ \Rightarrow \sum_{i=1}^{3} f_i^2(t) &= 1 \\ \Rightarrow \frac{\partial}{\partial t} \sum_{i=1}^{3} f_i^2(t) &= 0 \\ \Rightarrow 2 \sum_{i=1}^{3} f_i(t) f_i'(t) &= 0 \end{aligned}$$

Thus proven. □

19

Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved in terms of x, y, z respectively, but not u.

Proof. We write out the augmented matrix. In terms of x

$$\left(\begin{array}{ccc|c} 1 & -1 & u^2 & -3 \\ -1 & 2 & u & -1 \\ 2 & -3 & 2u & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & u^2 & -3 \\ 0 & -1 & -u^2+3 & 4 \\ 0 & 0 & -u^2+4u+3 & -4 \end{array}\right)$$

In terms of y

$$\left(\begin{array}{ccc|c} 3 & -2 & u^2 & -1 \\ 1 & 2 & u & 1 \\ 2 & -3 & 2u & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1/3 & u^2/3 & -1/3 \\ 1 & 2 & u & 1 \\ 0 & -7 & 0 & -4 \end{array}\right)$$

In terms of z

$$\left(\begin{array}{ccc|c} 3 & 1 & u^2 & 1 \\ 1 & -1 & u & -2 \\ 2 & 2 & 2u & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 1 & u^2 & 1 \\ 1 & -1 & u & -2 \\ 0 & 4 & 0 & 7 \end{array}\right)$$

In terms of u

$$\left(\begin{array}{ccc|c} 3 & 1 & -1 & -u^2 \\ 1 & -1 & 2 & u \\ 0 & 2 & -3 & 2u \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & u^2 \\ 1 & -1 & 2 & u \\ 0 & 4 & -7 & 0 \end{array}\right)$$

y has to be written in z or the other way around in order for the equation to be solved. □

24

$$f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$$

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x} &= (3x^2y - y^3)(x^2+y^2)^{-1} + (x^3y + xy^3)(-1)(x^2+y^2)^{-2}2x \\ &= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial f(x,y)}{\partial y} &= (x^3 - 2xy^2)(x^2+y^2)^{-1} + (x^3y - xy^3)(-1)(x^2+y^2)^2(2y) \qquad (1) \\ &= \frac{x^5 - 4x^3y2 + x^3y^2}{(x^2+y^2)^2} \qquad (2)\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial f(0,0)}{\partial x} &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = 0 \\ \frac{\partial f(0,0)}{\partial y} &= \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = 0\end{aligned}$$

and so

$$\begin{aligned}\frac{\partial f(0,0)}{\partial x \partial y} &= \lim_{h\to 0} \frac{\partial f(0,0+h)/\partial x - \partial f(0,0)/\partial x}{h} = -1 \\ \frac{\partial f(0,0)}{\partial y \partial x} &= \lim_{h\to 0} \frac{\partial f(0+h,0)/\partial x - \partial f(0,0)/\partial x}{h} = 1\end{aligned}$$

Thus the partial derivative exists but the the partial derivatives are different.