HW11

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Question 1

For $n = 2$, we define

$$\Omega_1 = |x|^{-2}(x_1 dx_2 - x_2 dx_1)$$

for $n = 3$ we define

$$\Omega_2 = |x|^{-3}(x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge x_3 + x_3 dx_1 \wedge x_2)$$

(a)

Can you prove $d\Omega_1 = 0$, $d\Omega_2 = 0$

Proof.

$$\begin{aligned}
\Omega_1 &= |x|^{-2}(x_1 dx_2 - x_2 dx_1) \\
&= (x_1^2 + x_2^2)^{-1}(x_1 dx_2) - (x_1^2 + x_2^2)^{-1}(x_2 dx_1) \\
\Rightarrow d\Omega_1 &= d(x_1^2 + x_2^2)x_1 \wedge dx_2 - d(x_1^2 + x_2^2)x_2 \wedge dx_1 \\
&= \{-(x_1^2 + x_2^2)^{-2} 2x_1^2 + (x_1^2 + x_2^2)^{-1}\} dx_1 \wedge dx_2 \\
&\quad -\{-(x_1^2 + x_2^2)^{-2} 2x_2^2 + (x_1^2 + x_2^2)^{-1}\} dx_2 \wedge dx_1 \\
&= \frac{x_1^2 - x_2^2}{(x_1 + x + 2)^2} dx_{12} - \frac{x_1^2 - x_2^2}{(x_1 + x + 2)^2} dx_{12} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Omega_2 &= |x|^{-3}(x_1 dx_2 \wedge x_3 + x_2 dx_1 \wedge x_3 + x_3 dx_1 \wedge x_2) \\
&= |x|^{-3} x_1 dx_2 \wedge x_3 + |x|^{-3} x_2 dx_1 \wedge x_3 + |x|^{-3} x_3 dx_1 \wedge x_2 \\
\to d\Omega_2 &= d\left(\frac{x_1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} dx_2 \wedge dx_3\right) \\
&\quad -d\left(\frac{x_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} dx_1 \wedge dx_3\right) \\
&\quad +d\left(\frac{x_3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} dx_1 \wedge dx_2\right) \\
&= \frac{x_2^2 + x_3^2 - 2x_1^2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} dx_{123} - \frac{x_1^2 + x_3^2 - 2x_2^2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} dx_{213} \\
&\quad \frac{x_1^2 + x_2^2 - 2x_3^2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} dx_{321} = 0 dx_{123} = 0
\end{aligned}$$

□

(b)

Can you write down the expression for general n? Can you prove $d\Omega_n = 0$ for the general case?

$$\Omega_n = |x|^{-n}\left(\sum_{i=1}^{n}(-1)^{i+1}x_i dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots dx_n\right) \tag{1}$$

$$= \sum_{i=1}^{n} \frac{x_i(-1)^{i+1}}{(\sum_{i=1}^{n} x_i^2)^{n/2}} dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots dx_n \tag{2}$$

Proof. We prove directly that $d\Omega_n = 0$. Let $n \in \mathbf{N}$, then

$$\begin{aligned} d\Omega_n &= d\sum_{i=1}^{n}\left(\frac{(-i)^{i+1}}{(\sum_{i=1}^{n} x_i^2)^{n/2}} + \left(\frac{-n}{2}\right)\frac{x_i(-1)^{i+1}\cdot 2x_i}{(\sum_{i=1}^{n} x_i^2)^{n/2+1}}\right) \wedge dx_1 \wedge \\ &\quad \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots dx_n \\ &= \sum_{i=1}^{n} \frac{(-1)^{i+1}((\sum_{j=1}^{n} x_j^2) - nx_i^2)}{(\sum_{j=1}^{n} x_j^2)^{n/2}} dx_i \wedge dx_1 \wedge \\ &\quad \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots dx_n \\ &= \sum_{i=1}^{n} \frac{(-1)^{2i}((\sum_{j=1}^{n} x_j^2) - nx_i^2)}{(\sum_{j=1}^{n} x_j^2)^{n/2}} dx_1 \wedge \ldots \wedge dx_n \\ &= 0 \end{aligned}$$

(c)

$\Omega_2 = |x|^{-3}x_1 d_2 \wedge d_3 - |x|^{-3}x_2 d_1 \wedge d_3 + |x|^{-3}x_3 d_1 \wedge d_2$

$$\begin{aligned} \frac{\partial\varphi_{23}}{\partial s\partial t} &= \det\begin{pmatrix} \cos(\pi s)\pi\cos(2\pi)t & -\sin(\pi s)\sin(2\pi t)2\pi \\ -\sin(\pi s)\pi & 0 \end{pmatrix} \\ &= -\sin^2(\pi s)\sin(2\pi t)2\pi \end{aligned}$$

$$\begin{aligned} \frac{\partial\varphi_{13}}{\partial s\partial t} &= \det\begin{pmatrix} \cos(\pi s)\pi\cos(2\pi)t & -\sin(\pi s)\sin(2\pi t)2\pi \\ -\sin(\pi s)\pi & 0 \end{pmatrix} \\ &= -\sin^2(\pi s)\sin(2\pi t)2\pi \end{aligned}$$

$$\begin{aligned} \frac{\partial\varphi_{12}}{\partial s\partial t} &= \det\begin{pmatrix} \cos(\pi s)\pi\cos(2\pi)t & -\sin(\pi s)\sin(2\pi t)2\pi \\ \cos(\pi s)\pi\cos(2\pi)t & -\sin(\pi s)\sin(2\pi t)2\pi \end{pmatrix} \\ &= 0 \end{aligned}$$

Note that $x_1^2 + x_2^2 + x_3^2 = (2\sin^2(\pi s)\cos^2(2\pi t)) + \cos^2(\pi s)$ then

$$\begin{aligned}
\Omega_2(\gamma) &= \int_0^1 \int_0^1 |x|^{-3} x_1 d_2 \wedge d_3 + \int_0^1 \int_0^1 |x|^{-3} x_2 d_1 \wedge d_3 \\
&= \int_0^1 \int_0^1 \frac{\sin(\pi s)\cos(2\pi t)}{((2\sin^2(\pi s)\cos^2(2\pi t)) + \cos^2(\pi s))^{3/2}} \left(-\sin^2(\pi s)\sin(2\pi t)2\pi\right) dsdt \\
&\quad - \int_0^1 \int_0^1 \frac{\sin(\pi s)\cos(2\pi t)}{((2\sin^2(\pi s)\cos^2(2\pi t)) + \cos^2(\pi s))^{3/2}} \left(-\sin^2(\pi s)\sin(2\pi t)2\pi\right) dsdt \\
&= 0
\end{aligned}$$

$\square$