HW12

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Show that if $\omega = f_1 \mathrm{d}x\mathrm{d}y + f_2 \mathrm{d}x\mathrm{d}z + f_3 \mathrm{d}y\mathrm{d}z$ is closed, then it is exact.

Proof. Since $\mathrm{d}\omega = 0$,

$$\frac{\partial f_1}{\partial z}\mathrm{d}x\mathrm{d}y\mathrm{d}z - \frac{\partial f_2}{\partial y}\mathrm{d}x\mathrm{d}y\mathrm{d}z + \frac{\partial f_3}{\partial z}\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$

so

$$f_{1z} - f_{2y} + f_{3z} = 0$$

Define

$$F_1 = \int_{t=0}^{t=1} f_1(tx, ty, tz)t\mathrm{d}t$$

$$F_2 = \int_{t=0}^{t=1} f_2(tx, ty, tz)t\mathrm{d}t$$

$$F_3 = \int_{t=0}^{t=1} f_3(tx, ty, tz)t\mathrm{d}t$$

Define $\rho(x, y, z, t) = (tx, ty, tz)$
Define $\beta = \sum_I f_I dx_I + \sum_J g_J dt \wedge dx_J$
Define $N = \sum_J \left(\int_0^1 g_J(x, t)dt \right) dx_J$
then $(dN + Nd)\beta = \sum_I (f_I(x, 1) - f_I(x, j0))dx_I$

Define $L = N \circ \rho^*$, and let $\omega = hdx_I \in \Omega^k(\mathbf{R}^n)$, then

$$\begin{aligned}
\rho^*(hdx_I) &= (\rho^* h)(\rho^*(dx_I)) \\
&= h(tx)d\rho_I \\
&= h(tx)(d(tx_{i1}) \wedge \ldots \wedge d(tx_{ik})) \\
&= h(tx)(t^k dx_I) + \text{terms that include } dt
\end{aligned}$$

So

$$(Nd + dN) \circ \rho^*(hdx_I) = hdx_I$$

so

$$(Ld + dL)(hdX_I) = hdx_I$$

so

$$(Ld + dL)\omega = \omega$$

□