# HW5

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## 8.2.7

Let $p > 2$ and $c > 0$. Using the Borel-Contelli lemma, show that the set

$$\{x \in [0,1] : |x - \frac{a}{q}| \leq \frac{c}{q^p} \text{ for infinitely many positive integers a,q}\}$$

has measure zero.

*Proof.* Let $\Omega_{p,a} = \{x \in [0,1] : |x - \frac{a}{q}| \leq \frac{c}{q^p}\}$. We want to show that $\sum_{q,a}^{\infty} \Omega_{q,a}$ is measurable, and $\sum_{q,a}^{\infty} m(\Omega_{q,a}) < \infty$. First we try to divide $q, a$ into cases. Suppose we fix $q$, then let

$$x \in \Omega_{p,a} \Rightarrow x \in \left[\frac{a}{q} - \frac{c}{q^p}, \frac{a}{q} + \frac{c}{q^p}\right] \tag{1}$$

Since both the bounds tends toward infinity, there exists some $N_q$ such that if $a > N_q$, $\Omega_{n,p} = \emptyset$. Also, there exists some $N$ such that if $q > N$ then $\frac{4c}{q^{p-1}} < 1$, suppose $a > 2q$, then $\frac{a}{q} - \frac{c}{q^p} > 1$, thus $\Omega_{q,a} = \emptyset$.
To summarize,

$$\begin{cases} \forall q \leq N, & a > N_q \Rightarrow \Omega_{q,a} = \emptyset \\ \forall q \leq N, & a \leq N_q \Rightarrow m(\Omega_{q,a}) \leq 1 \\ \forall q > N, & a > 2q \Rightarrow \Omega_{q,a} = \emptyset \\ \forall q > N, & a \leq 2q \Rightarrow m(\Omega_{q,a}) = \frac{2c}{q^p}. \end{cases}$$

Thus,

$$\sum_{q=1}^{\infty}\sum_{a=1}^{\infty} m(\Omega_{q,a}) = \sum_{q=1}^{N}\sum_{a=1}^{N_q} m(\Omega_{p,q}) + \sum_{q=N+1}^{\infty}\sum_{a=1}^{2q} m(\Omega_{q,a}) \tag{2}$$

$$\leq \infty \tag{3}$$

Thus by Borel-Contelli lemma, $m(\{x \in \Omega_{q,a} : a, q \in \mathbf{N}^+\}) = 0$ □

## 8.2.9

For every positive integer $n$, let $f_n : \mathbf{R} \to [0, \infty)$ be a non-negative measurable function such that

$$\int_{\mathbf{R}} f_n \leq \frac{1}{4^n}$$

Show that for every $\epsilon > 0$ there exists a set $E$ of Lebesgue measurable $m(E) \leq \epsilon$ such that $f_n(x)$ converges pointwise to zero for all $x \in \mathbf{R} \setminus E$.

*Proof.* Let $\epsilon > 0$. Let $A_n = \{x \in \mathbf{R} : f_n(x) > \frac{1}{\epsilon 2^n}\}$, define a simple function

$$s_n = \begin{cases} \frac{1}{\epsilon 2^n} & x \in A_n \\ 0 & \text{otherwise} \end{cases}$$

then $\int s_n = \frac{1}{\epsilon 2^n} m(A_n)$.
$\because s_n \leq f_n$ for all $n$, so

$$\int s_n \leq \int f_n \leq \frac{1}{4^n}$$
$$\to \frac{1}{\epsilon 2^n} m(A_n) \leq \frac{1}{4^n}$$
$$\to m(A_n) \leq \frac{\epsilon}{2^n}$$

□

Let $E = \bigcup_{n=1}^{\infty} A_n$,
$\because A_n$ is preimage of an open set in the range of a measurable function $f_n$,
$\therefore A_n$ is measurable, and so is $E = \bigcup_{n=1}^{\infty} A_n$ .

$$m(E) = m\left(\bigcup_{n=1}^{\infty} A_n\right)$$
$$\leq \sum_{n=1}^{\infty} m(A_n) = \epsilon$$

$E$ is a zero set. If $x \in R \setminus E$, then $f_n(x) \leq \frac{\epsilon}{2^n}$ True for all $n$, thus $f_n \to 0$.

## 8.2.10

For every positive integer $n$, let $f_n : [0,1] \to [0,\infty)$ be a non-negative measurable function such that $f_n$ converges pointwise to zero. Show that for every $\epsilon > 0$, there exists a set $E$ of Lebesgue measure $m(E) \leq \epsilon$ such that $f_n(x)$ converges uniformly to zero for all $x \in [0,1] \setminus E$

*Proof.* Let $m \in Z^+$, $A_n = \{x \in [0,1] | f_n(x) > \frac{1}{m}\}$, then let $B_n = \bigcup_{i=n}^{\infty} A_n$, then $B_1 \supset B_2 \supset \ldots$, so $m(\bigcap_{n=1}^{\infty} B_n) = \lim m^*(B_n)$. Claim that $\lim m^*(B_n) = \lim m^*(\emptyset) = 0$. Suppose otherwise, $\exists x \in \bigcap_{n=1}^{\infty} B_n \Rightarrow \forall n \quad x \in B_n$, but $f_n \to 0$, so there exists some $N$ such that $n > N$, then $f_n(x) < \frac{1}{m}$, $x \notin A_n$, a contradiction. Thus $\lim m^*(B_n) = \lim m^*(\emptyset) = 0$. So there $\exists N_m$ such that if $n > N_m$, then $m^*(B_n) \leq \frac{\epsilon}{2^m}$. Let $E = \bigcup_{m=1}^{\infty} B_{N_m}$, then $m(E) \leq \sum m(B_{N_m}) \leq \sum \frac{\epsilon}{2^m} = \epsilon$. Let $\epsilon > 0$, choose $M$ s.t. $\frac{1}{M} < \epsilon$. Suppose $x \in [0,1] \setminus E$, then $x \notin B_{N_M}$ and thus $x \notin A_n$ for all $n > N_M$. i.e. $f_n(x) \leq 1/M < \epsilon$. $f_n$ converges uniformly on $[0,1] \setminus E$.

No consider the function mentioned in class, the sequence of indicator functions lumping in $[n, n+1]$. does not converge uniformly to zero. □