# HW6

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## 8.3.2

### (a)

*Proof.*

$$\begin{aligned}\int_\Omega cf &= \int_\Omega cf^+ - \int_\Omega cf^- \\ &= c\int_\Omega f^+ - c\int_\Omega f^- \\ &= c\left(\int_\Omega f^+ - \int_\Omega f^-\right) \\ &= c\int_\Omega f\end{aligned}$$

□

### (b)

*Proof.* WTS

$$\int_\Omega f+g = \int_\Omega f + \int_\Omega g$$

But we know that

$$\int_\Omega f = \int_\Omega f^+ - \int_\Omega f^-$$

$$\int_\Omega g = \int_\Omega g^+ - \int_\Omega g^-$$

$$\int_\Omega f+g = \int_\Omega (f+g)^+ - \int_\Omega (f+g)^-$$

So it suffices to show

$$\int_\Omega (f+g)^+ - \int_\Omega (f+g)^- = \int_\Omega f^+ - \int_\Omega f^- + \int_\Omega g^+ - \int_\Omega g^-$$

rearranging terms

$$\int_\Omega (f+g)^+ + \int_\Omega f^- + \int_\Omega g^- = \int_\Omega (f+g)^- + \int_\Omega f^+ + \int_\Omega g^+$$

$$\int_\Omega (f+g)^+ + f^- + g^- = \int_\Omega (f+g)^- + f^+ + g^+$$

□

iff $(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$ almost everywhere. Thus we divide by cases
Let $x \in \Omega$ s.t. $f(x)+g(x) \geq 0$
if $f(x) > 0,\ g(x) > 0 \Rightarrow f(x)+g(x)+0+0 = 0+f(x)+g(x)$
if $f(x) > 0,\ g(x) < 0 \Rightarrow f(x)+g(x)-g(x)+0 = 0+f(x)+0$
if $f(x) < 0,\ g(x) > 0 \Rightarrow f(x)+g(x)-f(x)+0 = 0+0+g(x)$
if $f(x) < 0,\ g(x) < 0 \Rightarrow$ does not exist
Let $x \in \Omega$ s.t. $f(x)+g(x) < 0$
if $f(x) > 0,\ g(x) > 0 \Rightarrow$ does not exist
if $f(x) > 0,\ g(x) < 0 \Rightarrow 0+0+(-g(x)) = -(f(x)+g(x))+f(x)+0$
if $f(x) < 0,\ g(x) > 0 \Rightarrow 0+(-f(x))+g(x) = -(f(x)+g(x))+0+g(x)$
if $f(x) < 0,\ g(x) < 0 \Rightarrow 0+(-f(x))+(-g(x)) = -(f(x)+g(x))+0+0$ And thus $(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$ and so

$$\int_\Omega f+g = \int_\Omega f + \int_\Omega g$$

## (c)

*Proof.* If $f(x) \leq g(x)$ for all $x \in \Omega$ then

$$\begin{aligned}\int_\Omega g - \int_\Omega f &= \int_\Omega g^+ - \int_\Omega g^- - \int_\Omega f^+ + \int_\Omega f^- \\ &= \left(\int_\Omega g^+ - \int_\Omega f^+\right) + \left(\int_\Omega f^- - \int_\Omega g^-\right)\end{aligned}$$

$\because \min\{f,0\} \leq \min\{g,0\}$
$\therefore f^- = -\min\{f,0\} \geq -\min\{g,0\} = g^-$
$\therefore \int_\Omega f^- \geq \int_\Omega g^-$, and similarly, $\int_\Omega g^+ \geq \int_\Omega f^+$, thus it follows that

$$\int_\Omega g - \int_\Omega f \geq 0$$

□

## (d)

*Proof.* let $Z = \{x | g(x) \neq f(x)\}$ be a null set such that for any $x \in \Omega \setminus Z$, $f(x) = g(x)$. Then

$$\begin{aligned}\int_\Omega f &= \int_\Omega f^+ - \int_\Omega f^- \\ &= \int_{\Omega\setminus Z} f^+ + \int_Z f^+ - \int_{\Omega\setminus Z} f^- - \int_Z f^- \\ &= \int_{\Omega\setminus Z} f^+ - \int_{\Omega\setminus Z} f^-\end{aligned}$$

similarly, we can get

$$\int_{\Omega} g = \int_{\Omega \setminus Z} g^{+} - \int_{\Omega \setminus Z} g^{-}$$

By the hypothesis that $g = f$ a.e, we get $\int_{\Omega \setminus Z} g^{+} = \int_{\Omega \setminus Z} f^{+}$ and that $\int_{\Omega \setminus Z} g^{-} = \int_{\Omega \setminus Z} f^{-}$. Thus it follows that $\int_{\Omega} f = \int_{\Omega} g$ $\square$

## 8.3.3

*Proof.* let $f, g$ be absolutely integrable s.t. $f(x) \leq g(x)$ and $\int_{\mathbf{R}} f = \int_{\mathbf{R}} g$, WTS that $f(x) = g(x)$ a.e, that is, $\exists Z$ s.t. $x \in \mathbf{R} \setminus Z$ then $f(x) = g(x)$. Want to find $Z$. Let $Z = \{x | f(x) < g(x)\}$, suppose $m^*(Z) \geq 0$, then we decompose $\int_{\mathbf{R}} f$ into the form similar to part (d) in exercise 8.3.2

$$\begin{aligned}\int_{\mathbf{R}} f &= \int_{\mathbf{R}} f^+ - \int_{\mathbf{R}} f^- \\ &= \int_{\mathbf{R}\setminus Z} f^+ + \int_Z f^+ - \int_{\mathbf{R}\setminus Z} f^- - \int_Z f^-\end{aligned}$$

similarly, we can get

$$\int_{\mathbf{R}} g = \int_{\mathbf{R}\setminus Z} g^+ + \int_Z g^+ - \int_{\mathbf{R}\setminus Z} g^- - \int_Z g^-$$

But since $\int_{\mathbf{R}\setminus Z} g^+ = \int_{\mathbf{R}\setminus Z} f^+$, $\int_{\mathbf{R}\setminus Z} f^- = \int_{\mathbf{R}\setminus Z} g^-$, and that
$\int_Z g^+ > \int_Z f^+$, $\int_Z f^- > \int_Z g^-$ , thus $\int_{\mathbf{R}} f < \int_{\mathbf{R}} g$. A contradiction, thus $m(Z) = 0$. Thus for any $x \in \mathbf{R} \setminus Z$, $g(x) = f(x)$.
That is, $g(x) = f(x)$ a.e. □