HW7

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Ex 39

Suppose that f and g are measurable and their squares are integrable. Prove that fg is measurable, integrable, and

$$\int fg \leq \sqrt{\int f^2}\sqrt{\int g^2}$$

Proof. WTS $\int fg \leq \sqrt{\int f^2}\sqrt{\int g^2}$,
but since $\int fg \leq \int |fg|$,
thus, it suffices to show that $\int |fg| \leq \sqrt{\int f^2}\sqrt{\int g^2} \Leftrightarrow \int \frac{|fg|}{\sqrt{\int f^2}\sqrt{\int g^2}} \leq 1$.
First we claim that $\forall a, b \geq 0$, $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$. Since

$$\begin{aligned} ab &= \exp\{\ln(ab)\} \\ &= \exp\{\frac{1}{2}\ln(a)^2 + \frac{1}{2}\ln(b)^2\} \\ &\leq \frac{1}{2}\exp\{\ln(a)^2\} + \frac{1}{2}\exp\{\ln(b)^2\} \quad \text{by convexity of } e^x \\ &= \frac{a^2}{2} + \frac{b^2}{2} \end{aligned}$$

Now, set $a = \frac{|f(x)|}{\sqrt{\int f^2}}$, $b = \frac{|g(x)|}{\sqrt{\int g^2}}$ applying the claim, we get

$$\frac{|f(x)||g(x)|}{\sqrt{\int f^2}\sqrt{\int g^2}} \leq \frac{|f(x)|^2}{2\sqrt{\int f^2}} + \frac{|g(x)|^2}{2\sqrt{\int g^2}}$$

By taking integral of both sides

$$\int \frac{|f(x)||g(x)|}{\sqrt{\int f^2}\sqrt{\int g^2}} \leq \frac{1}{2} + \frac{1}{2} = 1$$

Thus

$$\int fg \leq \sqrt{\int f^2}\sqrt{\int g^2}$$

□

Ex 48

The Devil's ski slope. Recall from Chapter 3 that the Devil's staircase function $H : [0,1] \to [0,1]$ is continuous, nondecreasing constant on each interval complementary to the standard Cantor set, and yet is surjective. For $n \in \mathbf{Z}$ and $x \in [0,1]$ we define $\widehat{H}(x+n) = H(x) + n$. This extends H to a continuous surjection $\mathbf{R} \to \mathbf{R}$. Then we set

$$H_k(x) = \widehat{H}(3^k x) \quad \text{and} \quad J(x) = \sum_{k=0}^{\infty} \frac{H_k(x)}{4^k}$$

Prove that J is continuous, strictly increasing, and yet $J' = 0$ a.e.

Ex 53

Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$f(x,y) = \begin{cases} \frac{1}{y^2} & \text{if } 0 < x < y < 1 \\ \frac{-1}{x^2} & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a)

show that the iterated intergrals exist and are finite (calculated them) but the double integral does not exist.
The iterated integral

$$\begin{aligned} &\int_0^1 \int_0^x \frac{-1}{x^2} \mathrm{d}y \mathrm{d}x + \int_0^1 \int_x^1 \frac{1}{y^2} \mathrm{d}y \mathrm{d}x \\ =&\int_0^1 \frac{-1}{x} \mathrm{d}x + \int_0^1 -1 + \frac{1}{x} \mathrm{d}x \\ =&-1 < \infty \end{aligned}$$

Thus the integral exists. Similarly,

$$\begin{aligned} &\int_0^1 \int_0^y \frac{1}{y^2} \mathrm{d}x \mathrm{d}y + \int_0^1 \int_y^1 \frac{-1}{x^2} \mathrm{d}x \mathrm{d}y \\ =&\int_0^1 \frac{1}{y} \mathrm{d}y + \int_0^1 1 - \frac{1}{y} \mathrm{d}y \\ =&1 < \infty \end{aligned}$$

But the double integral does not exist.

(b)

Explain why (a) does not contradict Corollary 43.

Ex 58

The balanced density of a measurable set E at x is the limit, if exists, of the concentration of E in B where B is a ball centered at x that shrinks down to x. Write $\delta_{\text{balanced}}(x, E)$ to indicate the balanced density, and if it is 1, refer to x as a balanced density point.

(a)

Why is it immediate from the Lebesgue Density Theorem that almost every point of E is a balanced density point?
Since for every shrinking box, there is a smaller ball whose center is x, thus if by shrinking boxes, we get density 1 at x, x would have a balanced density of 1. So density equals to 1 everywhere implies balanced density equals to 1 everywhere.

(b)

Given $\alpha \in [0, 1]$, construct an example of a measurable set $E \subset R$ that contains a point x with $\delta_{\text{balanced}}(x, E) = \alpha$.

(c)

Given $\alpha \in [0, 1]$, construct an example of a measurable set $E \subset R$ that contains a point x with $\delta(x, E) = \alpha$. Here, we construct the set according to Real Analysis Exchange article:
First, we map out the proof outline:
Construct A_n, a infinite union of disjoint subintervals, get the measure closed form by countable additivity. Consider a smaller interval, such that the measure

Proof. Let $E = (a, b) \subset \mathbf{R}$, let $m = \frac{b-a}{2}$ let

$$A_n = \bigcup_{r=1}^{\infty} \left(a + \frac{nm}{n+r}, a + \frac{nm}{n+r} + \frac{\alpha nm}{(n+r)(n+r-1)} \right)$$

then for any positive integer N, by countable additivity

$$m\left(\bigcup_{r=N}^{\infty} \left(a + \frac{nm}{n+r}, a + \frac{nm}{n+r} + \frac{\alpha nm}{(n+r)(n+r-1)} \right) \right) = \alpha \left(\frac{nm}{n+N-1} \right)$$

And by such construct, we can find $c \in (a, a+m)$, and some $l \in \mathbf{N}$ such that $\frac{nm}{n+s+1} \leq m(A_n \cap (a, c)) \leq \frac{\alpha nm}{n+s}$ and $\frac{nm}{n+s+1} \leq (c-a) \leq \frac{nm}{n+s}$, thus the limit of the measure $\lim_{n\to\infty} \frac{m(A_n \cap (a,c))}{c-a}$ is bounded by two sequences that converges to α.

□

(d)

Is there a single set that contains points of both types of density for all $\alpha \in [0, 1]$?

Ex 66

Construct a monotone function $f : [0,1] \to \mathbf{R}$ whose discontinuity set is exactly the set $\mathbf{Q} \cap [0,1]$, or prove that such a function does not exist.

Proof. Let $Q = \{q_n \in [0,1] \setminus \mathbf{Q} | n \in \mathbf{N}\}$. Then let

$$f(x) = \sum_{q_n \le x} \frac{1}{n^2}$$

Then by the integral test, for every $x \in \mathbf{R}$, $f(x)$ converges absolutely. And for every $x' > x$, $f(x') > f(x)$, thus the function is monotone increasing. But for every element $q \in Q$,

$$\lim_{x \to q^-} f(x) = \sum_{q_n < q} \frac{1}{n^2} < \sum_{q_n \le q} \frac{1}{n^2} = \lim f(q)$$

And for $p \in [0,1] \setminus Q$,

$$\lim_{x \to p} f(x) = \sum_{q_n \le p} \frac{1}{n^2} = \lim f(p)$$

□