HW8

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1.

1. Littlewood's First Principle: every measurable set is the nearly the finite union of intervals. (differ by a set with measure less than ϵ, a.e. an $\epsilon-$set)
2. measurable functions are nearly continuous. (continuous given domain less $\epsilon-$set)
3. sequence of measurable functions are nearly uniformly convergence.

Measurable sets and functions have very nice properties suppose we are willing to discard some part of the domain/set. But it only says the existence, I was wondering if there is a systematic way of finding such $\epsilon-$sets for every epsilon. I think I have a similar question for a.e. as well, but since in that case, the set is measure zero, so by properties of null set, we could easily disregard their influence on the set as a whole.

2.

(a)

Proof. We apply similar argument to the $n-$dimensional case. Let

$$f_n : \prod_{i=1}^{p} (a_i, b_i) \to \mathbf{R}$$

and let

$$X(k,l) = \{x \in \prod_{i=1}^{p} (a_i, b_i) | \forall n \geq k \text{ we have } |f_k(x) - f(x)| < \frac{1}{l}\}.$$

Fix $l \in \mathbf{N}$. Since $f_n \to f$ a.e, $\bigcup_k X(k,l) \cup Z(l) = \prod_{i=1}^{p} (a_i, b_i)$.

Let $\epsilon > 0$, we know $\lim_{k\to\infty} m(X(k,l)) = \prod_{i=1}^{p} (a_i, b_i)$. Choose $k_1 < k_2 \dots$ such that for $X_l = X(k_l, l)$ we have $m(X_l^c) < \epsilon/2^l$. Let $X = \bigcap_l X_l$, we have $m(X^c) < \epsilon$.

Then f_n converges uniformly on X, since let $\sigma > 0$, choose l such $1/l < \sigma$. For all $n \geq k_l$ then $x \in X \Rightarrow x \in X_l = X(k_l, l) \Rightarrow |f_n(x) - f(x)| < 1/l < \sigma$ $\square$

(b)

True, consider f_n with unbounded domain in one dimension, define

$$X_m(k,l) = \{x \in [-m, m] | \forall n \geq k \text{ we have } |f_k(x) - f(x)| < \frac{1}{l}\}.$$

Let M be the finite measure of the unbounded domain. Then by measure continuity we get

$$\lim_{k\to\infty} \sup_m m(X_m(k,l) \cup Z(l)) = M$$

, thus we can choose $k_1, k_2, \dots,$ s.t. $X_l = \sup_m X_m(k_l, l)$ such that $m(X_l^c) < \epsilon/2^l$, then $m(X^c) < \epsilon$, where $X = \bigcap_l X_l$. Then let $\sigma > 0$, fix l such that $\frac{1}{l} < \sigma$, $n \geq k_l$. We get,

$$x \in X \Rightarrow x \in X_l = \sup_m X(k_l, l) \Rightarrow |f_n(x) - f(x)| < 1/l < \sigma.$$

Thus also works in unbounded finite measure domain.

(c)

Consider the classic moving bump function

$$f_n = \begin{cases} 1 & x \in [n, n+1] \\ 0 & \text{otherwise} \end{cases}$$

We show that the function does not converge uniformly less an ϵ-set. let $\epsilon = 0.5$ let $\sigma = 1$, let $N \in \mathbf{N}$, let $n > N$, choose $n \in [n, n+1] \backslash \epsilon-$set, then $f_n(x) - f(x) \geq \sigma$, true for all N, thus f_n does not converge uniformly even less an $\epsilon-$set.

(d)

Assume that the function converges pointwise on $\mathbf{R}^k$, a compact set K in $\mathbf{R}^n$ is closed and bounded. Bounded thus the domain has finite measure, and we can apply the Egoroff's Theorem directly. So for each compact set, we get

Little woods three principals refers to

3.

Proof. Recall the definition $||T||_1 = \sup_{x\neq 0} \frac{|Tx|_1}{|x|_1} = \sup_{|x|=1} |Tx|$, it suffices to show $\sup_{|x|=1} |Tx|$

$$\begin{aligned} |Tx| &= \left| \sum_{i=1}^{n} x_i T_{\cdot i} \right| \\ &\leq \sum_{i=1}^{n} |x_i T_{\cdot i}| \\ &= \sum_{i=1}^{n} |x_i||T_{\cdot i}| \end{aligned}$$

□

given the constraint that $|x| = 1$,

$$||T||_1 = \max |T_{\cdot i}|$$

, which is the column of T matrix with the greatest absolute sum.

4.

Outline:
Hölder's Inequality: By applying Young's Inequality for Products first, then apply the fact that if $\int f + g = 0$ then $f = g$ a.e. Then apply the iff equality condition for fullers inequality again.
Youngs inequality: if $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \frac{a^p}{p} + \frac{a^q}{q}$, equal iff $b = a^{p-1}$
Minkowski's inequality: decompose term on LHS, then apply the holder's inequality, then move terms.