HW9

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1.

If $f(0,0) = 0$ and

$$f(x,y) = \frac{xy}{x^2+y^2} \quad \text{if } (x,y) \neq (0,0)$$

prove that $(D_1 f)(x,y)$ and $(D_2)(x,y)$ exist at every point of $\mathbf{R}^2$, although f is not continuous at $(0,0)$.

Proof. Without loss of generality, we show that $(D_1 f)(x,y)$ is differentiable everywhere. Fix y, let $x \in \mathbf{R}$

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x} &= \frac{y(x^2+y^2)}{(x^2+y^2)^2} - \frac{2x^2y}{(x^2+y^2)^2} \\ &= \frac{y^3 - x^2y}{(x^2+y^2)^2}\end{aligned}$$

When $x = 0$, by L'Hospital's rule, $(D_1 f)(0,y)\big| = \frac{0}{2} = 0.$. Now we prove that f is not continuous at $(0,0)$. Set $\epsilon = \frac{1}{10}$, let $\delta > 0$, choose $z = (x_0, y_0) = (\delta/4, \delta/4)$, then $|z| < \delta$ and $|f(z) - f(0)| = \left|\frac{x_0 y_0}{x_0^2 + y_0^2}\right| = \frac{1}{2} < \frac{1}{10} = \epsilon$. This is true for all $\delta > 0$, thus f is not continuous at $(0,0)$. □

2.

Suppose that f is a real-valued function defined in an open set $E \subset \mathbf{R}^n$, and that the partial derivative $D_1 f, \ldots, D_n f$, are bounded in E. Prove that f is continuous in E.

Proof. Here we follow the hint in rudin. Let $\epsilon > 0$, let $x \in E$, and set $\delta = \frac{\epsilon}{2nM}$, let $M = \sup(\bigcup_i D_i f(E))$ let $\mathbf{h} = \sum h_j e_j$, $v_0 = 0$, $v_k = h_1 e_1 + \ldots + h_k e_k$, for $1 \leq k \leq n$, since E is open, we can choose $\mathbf{h}$, $|\mathbf{h}| < \delta$ such that $B_{|\mathbf{h}|}(x) \subset E$, then by construct, $x + h \in E$,$|x - (x + h)| = |h| < \delta$, and that

$$\begin{aligned} f(x+h) - f(x) &= \sum_{j=1}^{n} [f(x + v_j) - f(x + v_{j-1})] \\ \text{by the Mean Value Theorem} \\ &= \sum_{j=1}^{n} h_j (D_j f)(x + v_{j-1} + \theta_j h_j e_j) \\ \text{for some } \theta_j \in (0,1) \end{aligned}$$

Then

$$\begin{aligned} |f(x+h) - f(x)| &= |\sum_{j=1}^{n} h_j (D_j f)(x + v_{j-1} + \theta_j h_j e_j)| \\ &\leq |\sum_{j=1}^{n} h_j M| \quad M \text{ is the supremum} \\ &\leq \sum_{j=1}^{n} |h_j M| \quad \text{triangle inequality} \\ &\leq \sum_{j=1}^{n} |\mathbf{h}||M| = \epsilon/2 < \epsilon \end{aligned}$$

Thus f is continuous on E. □

3.

Show that, for any closed subset $E \subset \mathbf{R}^2$, there is a continuous function $f : \mathbf{R}^2 \to \mathbf{R}$, such that $f^{-1}(0) = E$.

Proof. The distance between a point and the set E satisfies the property.

$$d(x, E) = \inf\{d(x, y) | y \in E\}$$

It suffices to prove that $f : x \mapsto d(x, E)$ is continuous. Want to show:

$$\forall x_0 \in \mathbf{R}^2 \quad \forall \epsilon > 0 \quad \exists \delta > 0 \quad \text{s.t } \forall x \in \mathbf{R}^2 \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

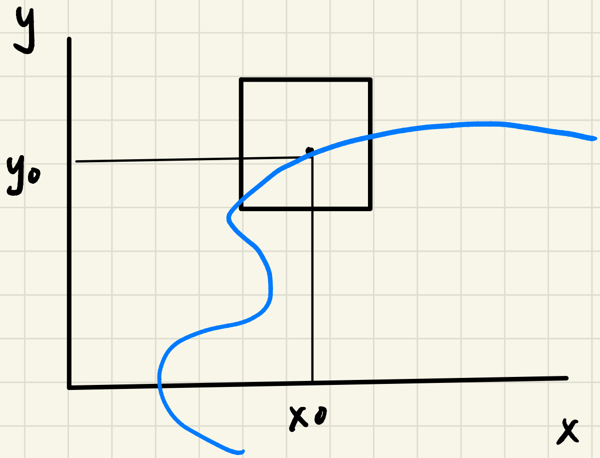
let $x_0 \in \mathbf{R}^2$, let $\epsilon > 0$, choose $\delta = \epsilon$. for every $x \in \mathbf{R}^2$, if $|x - x_0| < \delta$, we then want to show that $|f(x) - f(x_0)| < \epsilon$. It suffice to show that $|f(x) - f(x_0)| < \delta + \sigma$, for all $\sigma > 0$. We divide into cases, let $\sigma > 0$
1. suppose $f(x) > f(x_0)$, that is $d(x, E) > d(x_0, E)$, then there exists $y \in E$ such that $d(x_0, E) + \sigma > d(x_0, y)$. Then

$$\begin{aligned} d(x, y) - d(x_0, y) &< d(x, x_0) \\ \Rightarrow d(x, E) - d(x_0, y) &< d(x, x_0) \\ \Rightarrow d(x, E) - d(x_0, E) - \sigma &< d(x, x_0) \\ \Rightarrow d(x, E) - d(x_0, E) &< d(x, x_0) + \sigma \end{aligned}$$

True for all $\sigma > 0$, thus $d(x, E) - d(x_0, E) \leq d(x, x_0)$.
2. Similarly, suppose $f(x_0) > f(x)$, that is $d(x_0, E) > d(x, E)$, then there exists $y \in E$ such that $d(x, E) + \sigma > d(x, y)$. Then

$$\begin{aligned} d(x_0, y) - d(x, y) &< d(x, x_0) \\ \Rightarrow d(x_0, E) - d(x, y) &< d(x, x_0) \\ \Rightarrow d(x_0, E) - d(x, E) - \sigma &< d(x, x_0) \\ \Rightarrow d(x_0, E) - d(x, E) &< d(x, x_0) + \sigma \end{aligned}$$

True for all $\sigma > 0$, thus $d(x_0, E) - d(x, E) \leq d(x, x_0)$.
Thus for for any x_0, x, $|f(x) - f(x_0)| \leq d(x, x_0)$
$|f(x) - f(x_0)| \leq d(x, x_0) < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$ □



Intuition:

let (x_0, y_0) be the solution to

$f(x_0, y_0) = z_0$,

then, $\exists$ some radius r such that

$\forall (x, y) \in \mathbb{R}^n$ if $(x, y) \in B_r((x_0, y_0))$

the collection of such (x,y)'s can be written as function of x,

i.e $(x, g(x))$