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1 Some Reminder

Definition ($\sigma-$algebra)**.** $\sigma-$algebra is a collection of sets that includes the empty set, is closed under complement, and is closed under countable union.

1.1 Properties of open sets

- Countable union of open sets is open.
- Finite intersection of open sets is open.

1.2 Cartesian Products and sets

- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $(A \times B)^c = (A^c \times B) \cup (A \times B^c) \cup (A^c \times B^c)$

1.3 Cardinality

Refer to the following document:
https://www3.cs.stonybrook.edu/ cse547/definitions3.pdf

1.4 Intersections and Unions of Sets

- Distributive
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- Demorgan's law
  $(A \cup B)^c = A^c \cap B^c$
  $(A \cap B)^c = A^c \cup B^c$

1.5 Some topological concepts

Theorem 1.1 (Heine-Borel theorem)**.** *For $S \in \mathbf{R}^n$, S compact if and only if S is closed and bounded.*

Theorem 1.2 (Lebesgue's Number Lemma)**.** *Let* (X, d) *be a compact metric space. Then for every open cover* U *of* X*, there exists a number* $\delta > 0$*, such that every subset of* X *having diameter less than* δ *is covered in some member of the cover* U*.*

2 Outer Measure

Axiom (Desired Properties of Measurable sets)**.** A list of the nine properties

Measurability

(i) (Borel Property) Every open set in $\mathbf{R}^n$ is measurable, as is every closed set.

(ii) (Complementarity) If Ω is measurable, then $\mathbf{R}^n \setminus \Omega$ is also measurable.

(iii) (Boolean algebra property) If $(\Omega_j)_{j\in J}$ is any finite collection of measurable sets (so J is finite), then the union $\bigcup_{j\in J} \Omega_j$ and intersection $\cap_{j\in J}\Omega_j$ is also measurable.

(iv) (σ-algebra property) If $(\Omega_j)_{j\in J}$ is any countable collection of measurable sets (so J is countable), then the union $\bigcup_{j\in J} \Omega_j$ and intersection $\cap_{j\in J}\Omega_j$ is also measurable.

Other properties

(v) (Empty Set) The empty set $\emptyset$ has measure $m(\emptyset) = 0$.

(vi) (Positivity) We have $0 \leq m(\Omega) < +\infty$ for every measurable set Ω.

(vii) (Monotonicity) If $A \subseteq B$, and A and B are both measurable, then $m(A) \leq m(B)$.

(viii) (Finite sub-additivity) If $(A_j)_{j\in J}$ are a finite collection of measurable sets, then $m(\bigcup_{j\in J} A_j) \leq \sum_{j\in J} m(A_j)$.

(ix) (Finite additivity) If $(A_j)_{j\in J}$ are a finite collection of *disjoint* measurable sets, then $m(\bigcup_{j\in J} A_j) = \sum_{j\in J} m(A_j)$.

(x) (Countable sub-additivity) If $(A_j)_{j\in J}$ are a countable collection of measurable sets, then $m(\bigcup_{j\in J} A_j) \leq \sum_{j\in J} m(A_j)$.

(xi) (Countable additivity) If $(A_j)_{j\in J}$ are a countable collection of *disjoint* measurable sets, then $m(\bigcup_{j\in J} A_j) = \sum_{j\in J} m(A_j)$.

(xii) (Normalization) The unit cube $[0,1]^n = \{(x_1,\ldots,x_n) \in \mathbf{R}^n : 0 \leq x_j \leq 1 \text{ for all } 1 \leq j \leq n\}$ has measure $m([0,1]^n) = 1$.

(xiii) (Translation invariance) If Ω is a subset of $\mathbf{R}^n$, and $x \in \mathbf{R}^n$, then $x + \Omega := \{x + y : y \in \Omega\}$ $m^*(x+\Omega) = m^*(x)$

Definition (Open box)**.** An open box (or box for short) $B \in \mathbf{R}^n$ is any set of the form

$$B = \prod_{i=1}^{n}(a_i, b_i) := \{(x_1,\ldots,x_n) \in \mathbf{R}^n : x_i \in (a_i, b_i) \quad \text{for all } 1 \leq\}$$

where $b_i \geq a_i$ are real numbers. We define the volume $\text{vol}(B)$ of this box to be the number

$$\text{vol}(B) : \prod_{i=1}^{n}(b_i - a_i)$$

Definition (Outer measure)**.** If Ω is a set, we define the outer measure $m^*(\Omega)$ of Ω to be the quantity

$$m^*(\Omega) := \left\{ \sum_{j \in J} \text{vol}(B_j) : (B_j)_{j \in J} \text{ covers } \Omega; J \text{ at most countable} \right\}$$

Lemma 2.1 (Properties of outer measure)**.** *Outer measure satisfies the following properties*

(v) Empty set

(vi) Positivity

(vii) Monotonicity

(viii) Finite sub-additivity

(x) Countable sub-additivity

(xiii) Translation invariance

Proposition 2.1 (Outer measure of closed box)**.** *For any closed box*

$$B = \prod_{i=1}^{n}[a_i, b_i] := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_i \in [a_i, b_i] \textit{ for all } 1 \leq i \leq n\}$$

we have

$$m^*(B) = \prod_{i=1}^{n}(b_i - a_i)$$

Corollary 2.1. *For any open box*

$$B = \prod_{i=1}^{n}(a_i, b_i) := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_i \in (a_i, b_i) \textit{ for all } 1 \leq i \leq n\}$$

we have

$$m^*(B) = \prod_{i=1}^{n}(b_i - a_i)$$

In particular, outer measure obeys the normalization (xii).

Exercise 2.1. Let A be a subset of $\mathbf{R}^n$ and let B be a subset of $\mathbf{R}^m$. Then $m^*_{n+m}(A \times B) \leq m^*(A) \times m^*(B)$, in fact $m^*_{n+m}(A \times B) = m^*(A) \times m^*(B)$ (harder to prove).

Exercise 2.2.
(a) If $A_1 \subseteq A_2 \subseteq A_3 \ldots$ is an increasing sequence of measurable sets (i.e. $A_j \subseteq A_{j+1}$ for every positive integer j), then we have $m(\bigcup_{j=1}^{\infty} A_j) = \lim_{j\to\infty} m(A_j)$.
(b) If $A_1 \supseteq A_2 \supseteq A_3 \ldots$ is an decreasing sequence of measurable sets (i.e. $A_j \supseteq A_{j+1}$ for every positive integer j), then we have $m(\bigcap_{j=1}^{\infty} A_j) = \lim_{j\to\infty} m(A_j)$

Theorem 2.1 (Measure Continuity Theorem). *If $\{E_k\}$ and $\{F_k\}$ are sequences of measurable sets then*

upward measure continuity	$E_k \uparrow E \Rightarrow \omega E_k \uparrow E$
downward measure continuity	$F_k \downarrow F$ and $\omega F_1 < \infty \Rightarrow \omega F_k \downarrow \omega F$

3 Measurable sets

Definition (Measure Space)**.** A **measure space** is a triple (M,$\mathcal{M}$,μ) where $\mathcal{M}$ is a $\sigma-$algebra of subsets of M, and μ is a measure on $\mathcal{M}$. That is, $\mu : \mathcal{M} \to [0, \infty]$ has the three properties

(a) $\mu(\emptyset) = 0$

(b) μ is monotone: $A \subset B$ implies $\mu A \leq B$

(c) μ is countably additive on $\mathcal{M} : E = \bigsqcup E_i$ implies $\mu E = \sum \mu(E_i)$

Definition (Meseomorphism)**.** If $(M, \mathcal{M}, \mu)$, $(M', \mathcal{M}', \mu')$ then the mapping $T : \mathcal{M} \to \mathcal{M}'$ is a

mesemorphism if T sends each $E \in \mathcal{M}$ to $TE \in \mathcal{M}'$

meseomorphism if T is a bijection, and both T and T^{-1} are mesemorphism.

mesisometry if T is a meseomorphism and $\mu'(TE) = \mu E$ for each $E \in \mathcal{M}$. (This is also called **measure preserving transformation** and **isomorphism of measure spaces.**)

Theorem 3.1. *If a bijection increases outer measure by at most a factor of* t *and its inverse increases outer measure by at most a factor* $1/t$ *then it is a meseomorphism. If* $t = 1$ *it is a mesisometry.*

Definition (Abstract outer measure)**.** Let M be any set. The collection of all subsets of M is denoted as 2^M. An **abstract outer measure** on M is a function on $\omega : 2^M \to [0, \infty]$ that satisfies the three axioms of outer measure: $\omega(\emptyset) = 0$, ω is monotone, and ω is countably subadditive.

Definition (Lebesgue Measure)**.** Let E be a subset of $\mathbf{R}^n$. We say that E is *Lebesgue measurable* or *measurable* for short, iff we have the identity

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$

for every subset A of $\mathbf{R}^n$

Definition (Lebesgue Measure)**.** A set $E \subset \mathbf{R}^d$ is said to be *Lebesgue measurable* if, for every $\epsilon > 0$, there exists an open set $U \subset \mathbf{R}^d$ containing E such that $m^*(U \setminus E) \leq \epsilon$. If E is *Lebesgue measurable*, we refer to $m(E) := m^*(E)$ as the *Lebesgue measure* of E. (This quantity may be equal to $+\infty$). We also write $m(E)$ as $m^d(E)$ when we wish to emphasize the dimension d.

Lemma 3.1 (Half-spaces are measurable)**.** *The half-space*

$$\{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : x_j > 0\}$$

is measurable.

Remark. Half space of the form $\{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : x_j > 0\}$ or $\{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : x_j < 0\}$ for some $1 \leq j \leq n$ is measurable.

Lemma 3.2 (Properties of measurable sets)**.**

(a) If E is measurable, then $\mathbf{R}^n \setminus E$ is also measurable.

(b) Translation invariant

(c) Boolean algebra property

(d) Every open box, and every closed box, is measurable.

(e) Any set of outer measure zero (i.e. $m^*(E) = 0$) is measurable.

Lemma 3.3 (Finite additivity)**.** *If $(E_j)_{j\in J}$ are a finite collection of disjoint measurable sets and any set A (not necessarily measurable), we have*

$$m^* \left(A \cap \bigcup_{j\in J} E_j \right) = \sum_{j\in J} m^*(A \cap E_j)$$

Furthermore, we have $m(\bigcup_{j\in J} E_j) = \sum_{j\in J} m(E_j)$

Corollary 3.1. *If $A \subseteq B$ are two measurable sets, then $B \setminus A$ is also measurable, and*

$$m(B \setminus A) = m(B) - m(A)$$

Corollary 3.2 (Measure of closed box)**.** *The Lebesgue measure of a closed or partially closed box is the volume of its interior. The boundary of a box is a zero set.*

Lemma 3.4 (Countable additivity)**.** *If $(E_j)_{j\in J}$ are a countable collection of disjoint measurable sets , we have $m(\bigcup_{j\in J} E_j)$ is measurable, and $m(\bigcup_{j\in J} E_j) = \sum_{j\in J} m(E_j)$*

Lemma 3.5 ($\sigma-$algebra property)**.** *If $(\Omega_j)_{j\in J}$ is any countable collection of measurable sets (so J is countable), then the union $\bigcup_{j\in J} \Omega_j$ and intersection $\cap_{j\in J}\Omega_j$ is also measurable.*

Lemma 3.6. *Every open set can be written as a countable or finite union of open boxes.*

Lemma 3.7 (Borel property)**.** *Every open set, and every closed set is Lebesgue measurable.*

4 Regularity

Theorem 4.1. *Lebesgue measure is* ***regular*** *in the sense that each measurable set E can be sandwiched between an $F_\sigma-$set and a $G_\delta-$set, $F \subset E \subset G$, such that $G \setminus F$ is a zero set. Conversely, if there is such $F \subset E \subset G$ then E is measurable.*

Corollary 4.1. *A bounded subset $E \subset \mathbf{R}^n$ is measurable if and only if it has a* ***regularity sandwich*** *$F \subset E \subset G$ such that F is an $F_\sigma-$set, G is a $G_\delta-$set, and $mF = mG$.*

Lemma 4.1. *Every open set in $n-$space is a countable disjoint union of open cubs plus a zero set.*

Lemma 4.2. *Every open set is a countable disjoint union of balls plus a zero set.*

5 Affine Motions

Remark. An affine motion of $\mathbf{R}^n$ is an invertible linear transformation T followed by a translation.

Theorem 5.1. *An affine motion $T : \mathbf{R}^n \to \mathbf{R}^n$ is a meseomorphism. It multiplies measure by $|\det T|$.*

Theorem 5.2. *Every open set in $n-$space is a countable disjoint union of open cubes plus a zero set.*

Theorem 5.3. *Every open set is a countable disjoint union of balls plus a zero set.*

Inner Measures, Hulls, and Kernels

Definition (Hulls and Kernels)**.** Consider any bounded $A \in \mathbf{R}^n$, the infimum of the measure of open sets that contain A is achieved by a $G_\delta-$set called a **hull** denoted H_A. The inner measure of A is the supremum of the measure of closed sets that it contains, and is achieved by an $F_\sigma-$set called a **kernel** denoted K_A. We denote the inner measure of A m_*A.

Definition. The **measure theoretic boundary** of A is $\partial_m(A) = H_A \setminus K_A$.

Remark. $m(\partial_m(A)) = 0$

Theorem 5.4. *If $A \subset B \subset \mathbf{R}^n$ and B is a box then A is measurable if and only if it divides B cleanly.*

Remark. The theorem is also true for a bounded measurable set B instead of a box.

Lemma 5.1. *If A is contained in a box B then $mB = m_*(A) + m^*(B \setminus A)$*

6 Product and Slices

Definition (Null set)**.** A subset $E \subset S$ with $\omega(E) = 0$ is called the "zero set" or the "null set"

Lemma 6.1 (Property of null set)**.** *Let $E \subset S$ be a null set*

(1) if $E' \subset E$, then E' is a null set.

(2) for every $A \subset S$, $\omega(A \cup E) = \omega(A)$.

(3) for every $A \subset S$, $\omega(A \cap E^c) = \omega(A)$.

(4) $\omega(E) = 0$, then E is measurable.

(5) If Z is a null set, then F is measurable iff $F \cup Z$ is measurable.

Theorem 6.1 (Measurable Product Theorem)**.** *If $A \subset \mathbf{R}^n$ and $B \subset \mathbf{R}^k$ are measurable then $A \times B$ is measurable and*

$$m(A \times B) = mA \cdot mB.$$

By convention $0 \cdot \infty = \infty \cdot 0$

Lemma 6.2. *If A and B are boxes then $A \times B$ is measurable and $m(A \times B) = mA \cdot mB$*

Lemma 6.3. *If A or B is a zero set then so is $A \times B$.*

Lemma 6.4. *If U and V are open then $U \times V$ is measurable and $U \times V = mU \cdot mV$*

Definition (Slice)**.** A slice of $E \subset \mathbf{R}^n \times \mathbf{R}^k$ at $x \in \mathbf{R}^n$ is the set

$$E_x = \{y \in \mathbf{R}^k : (x, y) \in E\}$$

Theorem 6.2 (Zero Slice Theorem)**.** *If $E \subset \mathbf{R}^n \times \mathbf{R}^k$ is measurable then E is a zero set if and only if almost every slice of E is a zero set.*

Theorem 6.3 (Zero Slice Theorem)**.** *Let $Z = \{x \in \mathbf{R}^n | m_{\mathbf{R}^k}(E_x) \neq 0\}$ is measure zero in $\mathbf{R}^n$ then $m(E) = 0$.*

Lemma 6.5. *If $W \subset I^n \times I^k$, $x \in I^n$ is open and $X_\alpha = X_\alpha(W) = \{x : m(W_x) > \alpha\}$ then*

$$mW \geq m(X_\alpha(W)) \cdot \alpha$$

7 Lebesgue Integrals

Note: all discussions are about converging pointwise. a.e represents almost everywhere, which is up to a zero set.

Definition (Undergraph)**.** The undergraph of f is

$$\mathcal{U}f = \{(x,y) \in \mathbf{R} \times [0,\infty) : 0 \leq y < f(x)\}$$

Definition (Tao's measurable function)**.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}^m$ be a function. A function f is measurable iff $f^{-1}(V)$ is measurable for every open set $V \subset \mathbf{R}^m$.

Definition (Pugh's measurable function)**.** $f : \mathbf{R} \to (0,\infty)$ is measurable if $\mathcal{U}f \subset \mathbf{R}^2$ is measurable.

Definition (Lebesgue integrable)**.** Let f be a measurable function. The Lebesgue integral of the f is the measure of the undergraph.

$$\int f = m(\mathcal{U}f)$$

Definition. The function $f : \mathbf{R} \to [0,\infty)$ is **Lebesgue integrable** if its integral is finite. The set of integrable functions is denoted by L.

Theorem 7.1 (Monotone Convergence Theorem)**.** *Assume that (f_n) is a sequence of measruable functions $f_n : \mathbf{R} \to [0,\infty)$ and $f_n \uparrow f$ a.e as $n \to \infty$. Then*

$$\int f_n \uparrow \int f$$

Note that: $f_n \uparrow f$ is equivalent to $\lim_{n\to\infty} f_n = f$ and $f_n \leq f_{n+1}$.

Definition (Completed undergraph)**.** The completed undergraph of f is

$$\mathcal{U}f = \{(x,y) \in \mathbf{R} \times [0,\infty) : 0 \leq y \leq f(x)\}$$

Corollary 7.1. *If (f_n) is a sequence of integrable functions that converges monotonically downward to a limit function f almost everywhere then*

$$\int f_n \downarrow \int f.$$

Definition. If $f_n : X \to [0,\infty)$ is a sequence of functions then the lower and upper **envelope sequences** are

$$\underline{f}_n(x) = \inf\{f_k(x) : k \geq n\} \quad \overline{f}_n(x) = \sup\{f_k(x) : k \geq n\}$$

Proposition 7.1. $\mathcal{U}(\overline{f}_n) = \bigcup_{k\geq n} \mathcal{U}(f_k)$ *and* $\widehat{\mathcal{U}}(\underline{f}_n) = \bigcap_{k\geq n} \widehat{\mathcal{U}}(f_k)$

Theorem 7.2 (Dominated Convergence Theorem). *If* $f_n : \mathbf{R} \to [0, \infty)$ *is a sequence of measurable functions such that* $f - n \to f$ *a.e. and if there exists a function* $g : \mathbf{R} \to [0, \infty)$ *whose integral is finite and which is an upper bound for all the functions* f_n *then* f *is integrable and* $\int f_n \to \int f$ *as* $n \to \infty$

Corollary 7.2. *The pointwise limit of measurable functions is measurable.*

Theorem 7.3 (Fatou's Lemma). *If* $f_n :\to [0, \infty)$ *is a sequence of measurable functions then*

$$\int \liminf f_n \leq \liminf \int f_n$$