Math 105 HW 10

Rudin

(12) a) Let $\vec{p} = (p_1, p_2, p_3) \in K$. We have that

$$(\nabla f_1)(f^{-1}(\vec{p})) = \nabla f_1(x,y)$$
$$= (f_{1_s}(x,y), f_{1_t}(x,y))$$
$$= (-a\sin(x)\cos(y), -(b + a\cos(x))\sin(y))$$

Where $f(x,y) = \vec{p}$. To have $(\nabla f_1)(f^{-1}(\vec{p})) = 0$ we must thus have

(1) $\sin(x) = 0$ or $\cos(y) = 0$

(2) $\sin(y) = 0$ or $\cos(x) = -\frac{b}{a}$

Case 1: $\sin(x) = 0$

Then, $x = k\pi$ for some $k \in \mathbb{Z}$. This means $\cos(x) = \pm 1$. But since $a < b$, $\cos(x) \neq \frac{-b}{a}$, so we must have $\sin(y) = 0$, or $y = m\pi$ for some $m \in \mathbb{Z}$.

Case 2: $\cos(y) = 0$

Then, $y = k\pi + \frac{\pi}{2}$ for some $k \in \mathbb{Z}$. This means $\sin(y) \neq 0$, so we must have $\cos(x) = \frac{-b}{a}$ $\Rightarrow$ not possible since $0 < a < b$.

Now, we compute the possible values of $\vec{p}$.

For case 1:

k, m even:

$f(x,y) = (a+b, 0, 0)$

k even, m odd:
$f(x,y) = (-a-b, \quad 0, 0)$

k odd, m even:
$f(x,y) = (-a+b, \quad 0, 0)$

k, m odd:
$f(x,y) = (a-b, 0, 0)$

So there are exactly 4 points.

b) We have

$$\nabla f_3(x,y) = (a\cos(x), 0)$$

For this to equal 0 we want
$\cos(x) = 0 \iff x = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$.
$f(\frac{\pi}{2} + k\pi, y)$ is

$$\begin{cases} (b\cos(y), b\sin(y), a) & k \text{ even} \\ (b\cos(y), b\sin(y), -a) & k \text{ odd} \end{cases}$$

So this describes the possible values for $\vec{q}$.

c) $f_{1ss}(s,t) = -a\cos(s)\cos(t)$
$f_{1st}(s,t) = a\sin(s)\sin(t)$
$f_{1ts}(s,t) = a\sin(s)\sin(t)$
$f_{1tt}(s,t) = -(b+a\cos(s))\cos(t)$

For the second derivative test,

$$D = f_{1ss}f_{1tt} - (f_{1st})^2$$

$$= ab\cos(s)\cos(t)^2 + a^2\cos(s)^2\cos(t)^2 - a^2\sin(s)^2\sin(t)^2$$

For our four cases:

① k, m even
$D = ab + a^2 > 0$
$f_{1ss} = -a < 0$
$\rightsquigarrow$ local max

② k even, m odd
$D = ab + a^2 > 0$
$f_{1ss} = a > 0$
$\rightsquigarrow$ local min

③ k odd, m even
$D = -ab + a^2 < 0$
$\rightsquigarrow$ saddle point

④ k, m odd
$D = -ab + a^2 < 0$
$\rightsquigarrow$ saddle point

(13) Suppose $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$:

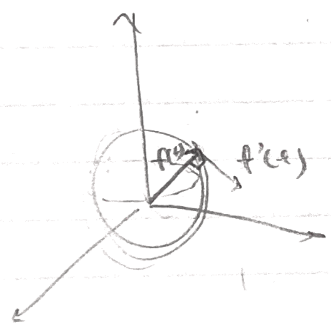
$|f(t)| = 1$ means $f_1(t)^2 + f_2(t)^2 + f_3(t)^2 = 1$

So taking the derivative wrt t,

$2f_1(t) \cdot f_1'(t) + 2f_2(t) \cdot f_2'(t) + 2f_3(t) f_3'(t) = 0$.

Dividing by 2 yields $\vec{f'}(t) \cdot \vec{f}(t) = 0$ as desired.

Geometric interpretation: The image of f is on the unit sphere, where the vector $\vec{f}(t)$ will always be orthogonal to the gradient.



(19)

$$\begin{aligned}
&(1) \quad 3x + y - z + u^2 = 0\\
&(2) \quad x - y + 2z + u = 0\\
&(3) \quad 2x + 2y - 3z + 2u = 0
\end{aligned}$$

$\underline{x, y, u \text{ in terms of } z:}$

$(1) + (2) \Rightarrow$

$$4x + z + u^2 + u = 0$$

$2 \cdot (2) + (3) \Rightarrow$

$$4x + z + 4u = 0$$

$\Rightarrow \quad u^2 - 3u = 0, \quad u = 0 \text{ or } 3$

$$x = \frac{-4u - z}{4}$$

$$y = u^2 + z - 3x$$

Solutions:

$$\begin{cases} u = 0 \\ x = -\frac{z}{4} \\ y = \frac{7}{4}z \end{cases} \quad \text{or} \quad \begin{cases} u = 3 \\ x = -3 - \frac{z}{4} \\ y = 18 + \frac{7}{4}z \end{cases}$$

$\underline{x, z, u \text{ in terms of } y:}$

· same operations give $u = 0$ or 3

$2 \cdot (1) + (2) \Rightarrow$

$$7x + y + 2u^2 + u = 0$$

$$x = \frac{-y - 2u^2 - u}{7}$$

$$z = 3x + y + u^2$$

Solutions:

$$\begin{cases} u=0 \\ x = \frac{-y}{7} \\ z = \frac{4y}{7} \end{cases} \qquad \begin{cases} u=3 \\ x = \frac{-y-21}{7} \\ z = \frac{4y}{7} \end{cases}$$

y, z, u in terms of x:

- $u = 0$ or 3
- from previous work, $z = -4x - 4u$

$$y = x + 2z + u$$

Solutions:

$$\begin{cases} u=0 \\ z=-4x \\ y=-7x \end{cases} \qquad \begin{cases} u=3 \\ z=-4x-12 \\ y=-7x-24 \end{cases}$$

x, y, z in terms of u:

The system is

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -u^2 \\ -u \\ -2u \end{bmatrix}$$

Since the first row is the sum of the other 2 but $-3u \neq -u^2$ necessarily there is no solution.

Pugh

(14) a) Not open.

Consider $A_\varepsilon = \begin{bmatrix} 1 & \varepsilon \\ 0 & 1 \end{bmatrix}$. ($\varepsilon > 0$). To find the eigenvalues we need to solve $\det(A - \lambda I) = 0$, or

$$\det \begin{bmatrix} 1-\lambda & \varepsilon \\ 0 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)^2 = 0$$
$$\lambda = 1$$

Note that

$$\begin{bmatrix} 1 & \varepsilon \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + \varepsilon x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

implies $x_2 = 0$, so the dimension of the eigenspace is 1. This implies that A_ε is not diagonalizable.

Thus since I is diagonalizable and ε is arbitrary we can get arbitrarily close to I with non-diagonalizable matrices, so the set of diagonalizable matrices is not open in $M(n \times n)$.

b) Not closed

A closed set contains all limit points.
Note that from a) the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is not diagonalizable. Consider

$$A_\varepsilon = \begin{bmatrix} 1 & 1 \\ \varepsilon^2 & 1 \end{bmatrix}$$

for $0 < \varepsilon < 1$. Solving for the eigenvalues:

$$\det \begin{bmatrix} 1-\lambda & 1 \\ \varepsilon^2 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)^2 - \varepsilon^2 = 0$$

$$1 - \lambda = \pm \varepsilon$$

$$\lambda = -1 \pm \varepsilon$$

Since there are two distinct eigenvalues, we can write a basis of eigenvectors that spans $\mathbb{R}^2$, so A_ε is diagonalizable. Taking $\varepsilon \to 0$ we approach A, but A is not diagonalizable. Therefore the set is not closed.

c) Not dense

A set $X \subseteq Y$ is dense if $\forall\ y \in Y,\ \varepsilon > 0$
$\exists\ x \in X$ s.t. $d(x, y) < \varepsilon$.

Consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 - 1 \\ \varepsilon_3 + 1 & \varepsilon_4 \end{bmatrix}$

Note that

$$\det \begin{bmatrix} \varepsilon_1 - \lambda & \varepsilon_2 - 1 \\ \varepsilon_3 + 1 & \varepsilon_4 - \lambda \end{bmatrix} = 0$$

$$\iff$$

$$(\varepsilon_1 - \lambda)(\varepsilon_4 - \lambda) - (\varepsilon_2 - 1)(\varepsilon_3 + 1) = 0$$

$$\iff$$

$$\lambda^2 - (\varepsilon_1 + \varepsilon_4)\lambda + \varepsilon_1\varepsilon_4 - \varepsilon_2\varepsilon_3 - \varepsilon_2 + \varepsilon_3 + 1 = 0$$

The discriminant of this quadratic is

$$(\varepsilon_1 + \varepsilon_4)^2 - 4[\varepsilon_1\varepsilon_4 - \varepsilon_2\varepsilon_3 - \varepsilon_2 + \varepsilon_3 + 1] =$$
$$[(\varepsilon_1 + \varepsilon_4)^2 - 4\varepsilon_1\varepsilon_4 + 4\varepsilon_2\varepsilon_3 + 4\varepsilon_2 - 4\varepsilon_3] - 4$$

As $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \to 0$, this approaches -4, so the quadratic has no solutions and thus B has no eigenvalues and is not diagonalizable. Therefore for sufficiently small ε,
$B(A, \varepsilon) \subseteq \{$ not diagonalizable 2×2 matrices $\}$
So the set of diagonalizable matrices must not be dense.

(24) Clearly all partials exist at points other than the origin, since f is a quotient of smooth functions and the denominator is non-zero. Thus we focus our attention to $(0,0)$. We have

$$\frac{\partial f(0,0)}{\partial x} = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h}$$

$$= 0$$

$$\frac{\partial f(0,0)}{\partial y} = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h}$$

$$= 0$$

(for $(x,y) \neq (0,0)$))

$$\frac{\partial f}{\partial x} = \frac{y(x^2-y^2) + 2x^2y}{x^2+y^2} - \frac{y(x^2-y^2)}{(x^2+y^2)^2 \cdot 2}$$

$$\frac{\partial f}{\partial y} = \frac{x(x^2-y^2) - 2xy^2}{x^2+y^2} - \frac{x(x^2-y^2)}{(x^2+y^2)\cdot 2}$$

$$\frac{\partial^2 f(0,0)}{\partial x^2} = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(h,0) - \frac{\partial f}{\partial x}(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h}$$

$$= 0$$

$$\frac{\partial^2 f(0,0)}{\partial y^2} = \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h}$$

$$= 0$$

$$\frac{\partial^2 f(0,0)}{\partial x \partial y} = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h}$$

$$= \lim_{h \to 0} \left(-\frac{h^3}{h^2} + \frac{h^3}{2h^4} \right) \frac{1}{h}$$

$$= \lim_{h \to 0} -1 + \frac{1}{2h^2}$$

$$= -1$$

$$\frac{\partial^2 f(0,0)}{\partial y \partial x} = \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h}$$

$$= \lim_{h \to 0} \left(\frac{h^3}{h^2} - \frac{h^3}{2h^4} \right) \frac{1}{h}$$

$$= \lim_{h \to 0} 1 - \frac{1}{2h^2} = 1$$