Math 105 HW 3

① Since a line is closed in $\mathbb{R}^2$ as it contains all its limit points, by HW 2 it is measurable. Thus, by the zero slice Theorem, $\{y=x\}$ is a zero set if every slice $\{y=x\}_x$, $x \in \mathbb{R}$ is measure 0. But these slices are points in $\mathbb{R}$, which have measure zero; so we are done.

② Theorem 16

We can express $E = \bigcup_{i=1}^{\infty} A_i$ where we define $A_i = E \cap (-i, i)^n$. Note that since $(-i,i)^n$ is bounded, so is each A_i, as $A_i \subset (-i, i)^n$.

Now if E is measurable, so is A_i, since it is the intersection of two measurable sets. Thus we may find F_σ and G_δ sets F_i, G_i for each i s.t. $F_i \subset A_i \subset G_i$, $m(G_i \setminus F_i) = 0$ by the bounded case. Then, $F = \bigcup_{i=1}^{\infty} F_i$ is an F_σ set and $G = \bigcap_{i=1}^{\infty} G_i$ is a G_δ set, and $F \subset E \subset G$.

We have $m(G \setminus F) = m\left(\bigcap_{i=1}^{\infty} G_i \setminus \bigcup_{i=1}^{\infty} F_i\right)$
$\leq m\left(\bigcup_{i=1}^{\infty} (G_i \setminus F_i)\right) \leq \sum_{i=1}^{\infty} m(G_i \setminus F_i) = 0$
by monotonicity and subadditivity. This generalizes the forward direction.

For the reverse, the argument from the textbook does not assume boundedness.

Theorem 21

For the unbounded case, we can write $E = \bigsqcup_{i=1}^{\infty} A_i$ by defining $A_1 = E \cap (-1,1)^n$ and for $i > 1$, $A_i = (E \cap (-i,i)^n) \setminus A_{i-1}$. Similarly, we can write $F = \bigsqcup_{i=1}^{\infty} C_i$. Note that the A_i, C_i are measurable, as they are finite intersections of measurable sets. Furthermore, they are bounded as $A_i \subseteq (-i,i)^n$, $C_i \subseteq (-i,i)^k$.

Thus note that $E \times F = \left(\bigsqcup_{i=1}^{\infty} A_i\right) \times \left(\bigsqcup_{i=1}^{\infty} C_i\right)$
$= \bigsqcup_{i=1}^{\infty} \bigsqcup_{j=1}^{\infty} A_i \times C_j$. Thus since by the finite case, each $A_i \times C_j$ is measurable, so since $E \times F$ is a countable union of measurable sets, it is measurable.

(3) First, we show $J^*A = J^*\bar{A}$. Since $A \subseteq \bar{A}$, monotonicity implies $J^*A \leq J^*\bar{A}$. For the reverse, let $\varepsilon > 0$. Let $\bigcup_{k=1}^{n} I_k$ be a finite covering of A by open intervals s.t. $\sum_{k=1}^{n} |I_k| < J^*A + \frac{\varepsilon}{2}$. For each $I_k = (a_k, b_k)$ define $I_k' = (a_k - \frac{\varepsilon}{4n}, b_k + \frac{\varepsilon}{4n})$. Note that $\bigcup_{k=1}^{n} I_k'$ covers $\bar{A}$, since $\bar{A} \subseteq \bigcup_{k=1}^{n} \bar{I_k} \subseteq \bigcup_{k=1}^{n} I_k'$. Furthermore, $\sum_{k=1}^{n} |I_k'| \leq \sum_{k=1}^{n} |I_k| + \frac{\varepsilon}{2}$, so $\sum_{k=1}^{n} |I_k'| < J^*A + \varepsilon$. Thus $J^*\bar{A} < J^*A + \varepsilon$ so since ε was arbitrary $J^*\bar{A} \leq J^*A$, showing $J^*A = J^*\bar{A}$.

To show that $m\bar{A} = J^*\bar{A}$, note that since $\bar{A}$ is closed and bounded, it is compact. Since closed sets are measurable, $m\bar{A}$ is well-defined and $m\bar{A} = m^*\bar{A}$. Clearly $m^*\bar{A} \leq J^*\bar{A}$ as finite coverings of $\bar{A}$ by open intervals are countable. To show the reverse direction, for any countable covering $\bigcup_{k=1}^{\infty} I_k$ of $\bar{A}$, by Heine-Borel we can find a finite sub-covering $\bigcup_{j\in J} I_j \subseteq \bigcup_{k=1}^{\infty} I_k$ (J is finite). Thus, since for each countable cover there is a finite subcover of $\leq$ length, we have that $J^*\bar{A} \leq m^*\bar{A}$. Thus $J^*A = J^*\bar{A} = m\bar{A}$ as desired.