

### Math 105 HW 3

① Since a line is closed in  $\mathbb{R}^2$  as it contains all its limit points, by HW 2 it is measurable. Thus, by the zero slice Theorem,  $\{y=x\}$  is a zero set if every slice  $\{y=x\}_x$ ,  $x \in \mathbb{R}$  is measure 0. But these slices are points in  $\mathbb{R}$ , which have measure zero, so we are done.

### ② Theorem 6b

We can express  $E = \bigcup_{i=1}^{\infty} A_i$  where we define  $A_i = E \cap (-i, i)^n$ . Note that since  $(-i, i)^n$  is bounded, so is each  $A_i$ , as  $A_i \subset (-i, i)^n$ .

Now if  $E$  is measurable, so is  $A_i$ , since it is the intersection of two measurable sets. Thus we may find  $F_0$  and  $G_0$  sets  $F_i, G_i$  for each  $i$  s.t.  $F_i \subset A_i \subset G_i$ ,  $m(G_i \setminus F_i) = 0$  by the bounded case. Then,  $F = \bigcup_{i=1}^{\infty} F_i$  is an  $F_0$  set and  $G = \bigcap_{i=1}^{\infty} G_i$  is a  $G_0$  set and  $F \subset E \subset G$ .

We have  $m(G \setminus F) = m\left(\bigcap_{i=1}^{\infty} G_i \setminus \bigcup_{i=1}^{\infty} F_i\right) \leq m\left(\bigcup_{i=1}^{\infty} (G_i \setminus F_i)\right) \leq \sum_{i=1}^{\infty} m(G_i \setminus F_i) = 0$  by monotonicity and subadditivity. This generalizes the forward direction.

For the reverse, the argument from the textbook does not assume boundedness.

### Theorem 21

For the unbounded case, we can

write  $E = \bigcup_{i=1}^{\infty} A_i$  by defining

$A_1 = E \cap (-1, 1)^n$  and for  $i > 1$ ,

$A_i = (E \cap (-i, i)^n) \setminus A_{i-1}$ . Similarly, we

can write  $F = \bigcup_{i=1}^{\infty} C_i$ . Note that the

$A_i, C_i$  are measurable, as they are finite intersections of measurable sets.

Furthermore, they are bounded as

$$A_i \subseteq (-i, i)^n, \quad C_i \subseteq (-i, i)^k.$$

Thus note that  $E \times F = \left( \bigcup_{i=1}^{\infty} A_i \right) \times \left( \bigcup_{j=1}^{\infty} C_j \right)$

$$= \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} A_i \times C_j.$$

Thus since by the finite case, each  $A_i \times C_j$  is measurable,

so since  $E \times F$  is a countable union of measurable sets, it is measurable.

$$J^*A + \epsilon \geq J^*\bar{A}$$

(3) First, we show  $J^*A = J^*\bar{A}$ . Since  $A \subseteq \bar{A}$ , monotonicity implies  $J^*A \leq J^*\bar{A}$ . For the reverse, let  $\epsilon > 0$ . Let  $\bigcup_{k \in \mathbb{N}} I_k$  be a finite covering of  $A$  by open intervals s.t.  $\sum_{k=1}^n |I_k| < J^*A + \frac{\epsilon}{2}$ . For each  $I_k = (a_k, b_k)$  define  $I'_k = (a_k - \frac{\epsilon}{4n}, b_k + \frac{\epsilon}{4n})$ . Note that  $\bigcup_{k=1}^n I'_k$  covers  $\bar{A}$ , since  $\bar{A} \subseteq \bigcup_{k=1}^n I_k \subseteq \bigcup_{k=1}^n I'_k$ . Furthermore,  $\sum_{k=1}^n |I'_k| \leq \sum_{k=1}^n |I_k| + \frac{\epsilon}{2}$ , so  $\sum_{k=1}^n |I'_k| \leq J^*A + \epsilon$ . Thus  $J^*\bar{A} < J^*A + \epsilon$  so since  $\epsilon$  was arbitrary  $J^*\bar{A} \leq J^*A$ , showing  $J^*A = J^*\bar{A}$ .

To show that  $m\bar{A} = J^*\bar{A}$ , note that since  $\bar{A}$  is closed and bounded, it is compact. Since closed sets are measurable,  $m\bar{A}$  is well-defined and  $m\bar{A} = m^*\bar{A}$ . Clearly  $m^*\bar{A} \leq J^*\bar{A}$  as finite coverings of  $\bar{A}$  by open intervals are countable. To show the reverse direction, for any countable covering  $\bigcup_{k \in \mathbb{N}} I_k$  of  $\bar{A}$ , by Heine-Borel we can find a finite sub-covering  $\bigcup_{j \in J} I_j \subseteq \bigcup_{k \in \mathbb{N}} I_k$  ( $J$  is finite). Thus, since for each  $\epsilon > 0$  countable covering is a finite sub-cover of  $\leq$  length, we have that  $J^*\bar{A} \leq m^*\bar{A}$ . Thus  $J^*A = J^*\bar{A} = m\bar{A}$  as desired.

① Proof #2

Consider  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$T(x, y) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Note that if  $(x, 0)$  is on the line  $y=0$ ,  $T(x, 0) = (x, x)$ . Thus, since  $T$  is invertible, it maps  $\{y=0\}$  to  $\{y=x\}$ . Thus, since  $m(\{y=0\}) = 0$ ,  $m(\{y=x\}) = |\det T| m(\{y=0\}) = 0$  as desired.

① Proof #3

A line has length  $0$ . Since  $\mathbb{R}^2$  is covered by a countable union of lines,  $\mathbb{R}^2$  is covered by a countable union of sets of length  $0$ . Thus,  $\mathbb{R}^2$  has length  $0$ .