Math 105 HW 4

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a) If f is measurable, $\mathcal{U}f$ and $\hat{\mathcal{U}}f$ are measurable, and $\hat{\mathcal{U}}f \setminus \mathcal{U}f$ is the graph of f. Thus, $\hat{\mathcal{U}}f \setminus \mathcal{U}f$ is measurable. Now, since any slice of a graph is a point, which has measure 0, by the Zero Slice Theorem, $m(\hat{\mathcal{U}}f \setminus \mathcal{U}f) = 0$.

b) No. Let $S \subseteq \mathbb{R}$ be a non-measurable set, and let I_S be the indicator function for S. Then, the graph of I_S is a zero set, as it is a subset of the line $y=1$ which has measure 0. However, clearly $\mathcal{U}I_S$ is not measurable (If it were, we could find G_δ + F_σ sets G, F s.t. $F \subset \mathcal{U}I_S \subset G$, $m(G \setminus F) = 0$, but the projecting onto $y=1$ shows that S is measurable, a contradiction.)

d) Every vertical slice of the stackexchange construction is a zero set, since this is a point. However, the function was constructed so that the graph has positive outer measure. Thus, the result of the zero-slice theorem relies on measurability, since this is the only condition violated.

e) AFSOC that we have a graph w/ positive inner measure. Then, $\exists$ a closed set K contained in the graph s.t. $m(K) > 0$. Thus, by the zero slice theorem, $\exists$ some x s.t. $m(K_x) > 0$. Thus, since a point has measure 0, K_x contains at least two points (x, y_1) and (x, y_2). But as K is contained in the graph of a function, and a function can take on one y value per x value, this is a contradiction so a graph cannot have positive inner measure. (K is measurable since it is closed).

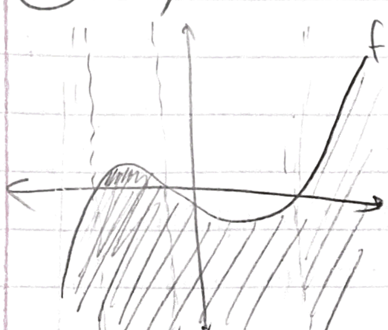
f) Let f be the function constructed in c). Suppose $m(\text{graph}(f)) > 0$. For each $r \in [0,1)$ define

$$E_r = \{ \text{graph}(f + n + r) \mid n \in \mathbb{Z} \}$$

Note that the E_r are disjoint, and there are uncountably many of them. Furthermore, since each E_r contains infinitely many disjoint sets of positive outer measure (translation invariance), it has infinite outer measure. Thus, we have an example of uncountably many disjoint subsets of the plane w/ $m^* = \infty$.

g) By taking the pre-image of the disjoint subsets of f, we can get uncountably many disjoint subsets of $\mathbb{R}$ w/ infinite outer measure.

(28) a)



Suppose $\mathcal{U}f$ is measurable. Let f^+, f^- respectively denote the positive and negative parts of f. Note that

$$\mathcal{U}f^+ = \mathcal{U}f \cap \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$$
$$\mathcal{U}f^- = \mathcal{U}f^c \cap \{(x,y) \in \mathbb{R}^2 : y \leq 0\}$$

Since half-planes are measurable, this shows that f^+, f^- are measurable.

Now, suppose f^+, f^- are measurable, then note that

$$\mathcal{U}f = (\mathcal{U}f^+ \cup \{(x,y) \in \mathbb{R}^2 : y \leq 0\}) \cap (\mathcal{U}f^-)^c$$

So $\mathcal{U}f$ is measurable.

b) By exercise 23, T is a mesomorphism. Thus, since

$(x,y) \in \mathcal{U}f \Leftrightarrow (x, \frac{1}{y}) \in (\mathcal{U}\frac{1}{f})^c \cap \{(x,y): y \geq 0\}$

as $0 \leq y < f(x) \Leftrightarrow \frac{1}{y} > \frac{1}{f(x)} > 0$

we have that $(\mathcal{U}\frac{1}{f})^c \cap \{(x,y): y \geq 0\}$ is measurable. Thus, since

$\mathcal{U}\frac{1}{f} = ((\mathcal{U}\frac{1}{f})^c \cap \{(x,y): y \geq 0\})^c \cap \{(x,y): y \geq 0\}$

$\mathcal{U}\frac{1}{f}$ is measurable.

c) Note that T is a mesomorphism because it is a diffeomorphism.

Since $\log(fg) = \log(f) + \log(g)$, $\log(fg)$ is measurable since $\mathcal{U}(\log f)$, $\mathcal{U}(\log g)$ are measurable, because $T(\mathcal{U}f) = \mathcal{U}(\log f)$, $T(\mathcal{U}g) = \mathcal{U}(\log g)$. Thus since $T^{-1}(\mathcal{U}(\log fg)) = \mathcal{U}(fg)$ and T^{-1} is a mesomorphism fg is measurable.

d) Given $X \subseteq \mathbb{R}$ we can repeat the same arguments but restricted to $X \subseteq \mathbb{R}$. This won't impact the validity of the proofs.

e) We can partition $\mathbb{R}$ into 4 regions:

$$X_1: \{x \mid f(x), g(x) > 0\}$$
$$X_2: \{x \mid f(x) > 0, \ g(x) < 0\}$$
$$X_3: \{x \mid f(x) < 0, \ g(x) < 0\}$$
$$X_4: \{x \mid f(x) < 0, \ g(x) > 0\}$$

Then, we apply the argument in c to fg restricted to $X_1 + X_3$ and $-fg$ restricted to $X_2 + X_4$. Thus we have shown that both the positive and negative parts of fg are measurable, so fg is measurable.