Math 105 HW 5

8.2.7

First, note that for $a > q$, $\frac{a}{q} > 1$ implies that for $x \in [0,1]$, $|x - \frac{a}{q}| > \frac{a}{q} - 1 = \frac{a-q}{q}$. Then, since $p > 2$, for $q \geq c$, $\frac{c}{q^p} < \frac{c}{q^2} \leq \frac{1}{q}$. Thus, since $a - q > 0$, and $a, q \in \mathbb{Z}$, $|x - \frac{a}{q}| > \frac{a-q}{q} \geq \frac{1}{q} > \frac{c}{q^2}$. Thus, since c is fixed and finite,

$$\{a, q \in \mathbb{Z}^+, a > q : |x - \frac{a}{q}| \leq \frac{c}{q^p}\}$$

is finite for each $x \in [0,1]$. Thus, we may consider just the cases where $0 \leq a \leq q$.

Now, define for $a, q \in \mathbb{Z}^+$

$$\Omega_{a,q} = \{x \in [0,1] : |x - \frac{a}{q}| \leq \frac{c}{q^p}\}$$

Note that $\Omega_{a,q} = [0,1] \cap [-\frac{c}{q^p} + \frac{a}{q}, \frac{c}{q^p} + \frac{a}{q}]$ so it is measurable. Thus, $m(\Omega_{a,q}) \leq \frac{2c}{q^p}$.

Since we consider just the cases where $0 \leq a \leq q$, note that for fixed q,

$$\sum_{a=0}^{q} m(\Omega_{a,q}) = \frac{2c(q+1)}{q^p}$$

Then, summing over all

$$\sum_{q=1}^{\infty} \sum_{a=0}^{q} m(\Omega_{a,q}) = \sum_{q=1}^{\infty} \frac{2c(q+1)}{q^p} = 2c\left(\sum_{q=1}^{\infty} \frac{1}{q^{p-1}} + \sum_{q=1}^{\infty} \frac{1}{q^p}\right)$$

which is finite because $p > 2$ implies $p - 1 > 1$, so both sums converge by the Integral Test.

Thus, by Borel-Cantelli, $\{x \in \mathbb{R} : x \in \Lambda_{a,q}$ for infinitely many $a, q\}$ has measure zero. By our definition of $\Lambda_{a,q}$, this proves the claim.

8.2.9

Let $\varepsilon > 0$. For $n \in \mathbb{N}$, let $E_n = \{x \in \mathbb{R} : f_n(x) > \frac{1}{\varepsilon 2^n}\}$. Since $\{y \in \mathbb{R} : y > \frac{1}{\varepsilon 2^n}\}$ is open, by the definition of a measurable function E_n is measurable.

We claim $m(E_n) \le \frac{\varepsilon}{2^n}$. Assume not. Then,
$\int_{E_n} f_n \ge \int_{E_n} \frac{1}{\varepsilon 2^n} = m(E_n) \cdot \frac{1}{\varepsilon 2^n} > \frac{1}{4^n}$

which is a contradiction, since $\int_{E_n} f_n \le \int_{\mathbb{R}} f_n \le \frac{1}{4^n}$.
Thus $m(E_n) \le \frac{\varepsilon}{2^n}$.

Now, let $E = \bigcup_{n=1}^{\infty} E_n$. By subadditivity,
$m(E) \le \sum_{n=1}^{\infty} m(E_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$. Let $x \in \mathbb{R} \setminus E$.
For $n \in \mathbb{N}$, $f_n(x) \le \frac{1}{\varepsilon 2^n}$. Thus, taking $n \to \infty$ on both sides, $\lim_{n\to\infty} f_n(x) \le 0$ so since f_n is non-negative, $f_n(x)$ converges to 0, showing the result.

8.2.10

Let $m \in \mathbb{N}$, $\varepsilon > 0$ and $S_N = \{x \in [0,1] : f_n(x) \leq 1/m \ \forall n \geq N\}$. Since f_n converges pointwise to zero, $\lim_{N\to\infty} S_N = [0,1]$. Thus since $S_N \subseteq S_{N+1}$ for each N, by HW1 $\lim_{n\to\infty} m(S_N) = m([0,1]) = 1$.

Now, note that:

$[0,1] \setminus S_N = \{x \in [0,1] : f_n(x) > 1/m$ for some $n \geq N\}$

and $m(S_N) + m([0,1] \setminus S_N) = 1$.

By the previous, $\exists N_m$ s.t.

$m([0,1] \setminus S_{N_m}) \leq \varepsilon/2^m$

Hence we have that

$m(\{x \in [0,1] : f_n(x) > 1/m$ for some $n \geq N_m\})$
$\leq \varepsilon/2^m$

Let $E = \bigcup_{m=1}^{\infty} \{x \in [0,1] : f_n(x) > 1/m$ for some $n \geq N_m\}$.
By countable subadditivity, $m(E) \leq \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon$.

Now, $\forall x \in [0,1] \setminus E$, $n > N_m$ implies $f_n(x) \leq \frac{1}{m}$. Since f_n is non-negative, this shows that on $[0,1] \setminus E$, $|f_n(x)| \leq \frac{1}{m}$. Taking $m \to \infty$, we see that $f_n \to 0$ uniformly on $[0,1] \setminus E$.

Note: $[0,1] \setminus S_N$ is measurable because it is the preimage of an open set, and hence so is S_N.

For the case where $[0,1]$ is replaced by $\mathbb{R}$, this is not true, as we can use the "moving bump" example. Let $f_n(x) = 1$ for $x \in [n, n+1)$ and 0 elsewhere. Then, for fixed x, $\lim_{n \to \infty} f_n(x) = 0$. But f_n cannot converge uniformly to 0, as $\forall N \in \mathbb{N}$, $|f_{N+1}| = 1$ on $[N+1, N+2)$.