Math 105 HW 7

(39) First, note that fg is measurable by Exercise 28.

Then, note that the set of measurable functions over $\mathbb{R}$ forms a vector space, as the constant function 0 is measurable, the sum of measurable functions is measurable, scaling by a constant does not affect measurability, and the remaining axioms follow from usual properties of functions.

Then, the inner product defined by $\langle f, g \rangle = \int fg$ is a valid inner product because $\int fg = \int gf$, $\int (af + bg)h = a\int fh + b\int gh$ (Thm 34), and $\int f^2 \geq 0$.

Thus the result follows by Cauchy-Schwarz.

(48) For $x = \sum_{i=1}^{\infty} \frac{w_i}{3^i}$ we have that
$\hat{H}(3^k x) = \hat{H}\left(\sum_{i=1}^{\infty} \frac{w_i}{3^{i-k}}\right) = \sum_{i=1}^{k} 3^{k-i} w_i + H\left(\sum_{i=k+1}^{\infty} \frac{w_i}{3^{i-k}}\right)$
$\leq 3^k$, so $J(x) \leq \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k$ and $J(x)$ is finite.

Thus, by the Weierstrass M-test, $J(x)$ converges uniformly (and absolutely). Thus since each $\hat{H}(3^k x)$ is continuous, J is continuous. J is strictly increasing because each H_k is strictly increasing.

Claim: $J'(x)$ exists almost everywhere.
For $x \in \mathbb{R}$:

$$\lim_{y \to x} \frac{\sum_{k=0}^{\infty} \frac{H_k(x)}{4^k} - \sum_{k=0}^{\infty} \frac{H_k(y)}{4^k}}{x - y} =$$

$$\lim_{y \to x} \sum_{k=0}^{\infty} \frac{H_k(x) - H_k(y)}{4^k (x-y)} =$$

$$\sum_{k=0}^{\infty} \frac{H_k'(x)}{4^k}$$

As long as x is not in the Cantor set. Then, since $H_k'(x) = \hat{H}'(3^k x) \cdot 3^k = H'(3^k x) \cdot 3^k = 0$ the limit is zero everywhere except the Cantor set points. Since the Cantor set has measure zero this completes the proof. (Still need uniform convergence to swap sum and limit though, not sure how to achieve this)

(53) a) We have for the iterated integrals

$$\int\left[\int f_x(y)\,dy\right]dx = \int_0^1\left[\int_x^1 \frac{1}{y^2}\,dy + \int_0^x -\frac{1}{x^2}\,dy\right]dx$$

$$= \int_0^1 \left[-\frac{1}{y}\right]_x^1 + \left(-\frac{1}{x}\right) \quad dx$$

$$= \int_0^1 -1 + \frac{1}{x} - \frac{1}{x}\,dx$$

$$= \boxed{-1}$$

$$\int\left[\int f_y(x)\,dx\right]dy = \int_0^1\left[\int_0^y \frac{1}{y^2}\,dx + \int_y^1 -\frac{1}{x^2}\,dx\right]dy$$

$$= \int_0^1 \frac{1}{y} + \left[\frac{1}{x}\right]_y^1 \quad dy$$

$$= \int_0^1 \frac{1}{y} + 1 - \frac{1}{y}\,dy$$

$$= \boxed{1}$$

Since f is non-negative, it is integrable if $f_\pm$ are integrable. We have

$$f_+ = \begin{cases} \frac{1}{y^2} & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_- = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

However, $f_\pm$ are not integrable. We show this for f_+ by constructing a set of disjoint boxes contained in $m^*(\mathcal{U}f)$ that have volume approaching infinity. For $n \in \mathbb{N}$, define $B_1, \dots B_n$ by

$$B_i = \left(0, \frac{i-1}{n}\right) \times \left(\frac{i-1}{n}, \frac{i}{n}\right) \times \left(0, \left(\frac{n}{i}\right)^2\right)$$

Note that the B_i are disjoint because the $\left(\frac{i-1}{n}, \frac{i}{n}\right)$ are disjoint intervals. They are contained in $\mathcal{U}f$ because for each $(x, y, z) \in B_i$, $0 < x < y < 1$, and as $y < \frac{i}{n}$, $f(x,y) = \frac{1}{y^2} > \left(\frac{n}{i}\right)^2$.

Furthermore, $\mathrm{Vol}(B_i) = \frac{i-1}{n} \cdot \frac{1}{n} \cdot \left(\frac{n}{i}\right)^2 = \frac{i-1}{i^2} = \frac{1}{i} - \frac{1}{i^2}$

So $\sum_{i=1}^{n} \mathrm{Vol}(B_i) = \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{n} \frac{1}{i^2}$. As $n \to \infty$, this sum goes to ∞, so $m^*(\mathcal{U}f) = \infty$ and f^+ is not integrable. Therefore the double integral does not exist.

b) Corollary 43 assumes that f is non-negative, which is not the case here.

(58) a) The Lebesgue Density Theorem says that almost every point of E is a density point. Every cube containing x (not equal to x) contains a ball containing x, so a.e. point of E is also a balanced density point.

b) For each $n \in \mathbb{N}$, define E_n^+ to be any subinterval of $\left(\frac{1}{n+1}, \frac{1}{n}\right)$ of length $\alpha\left(\frac{1}{n} - \frac{1}{n+1}\right)$ and E_n^- to be any subinterval of $\left(-\frac{1}{n}, -\frac{1}{n+1}\right)$ of length $\alpha\left(\frac{1}{n} - \frac{1}{n+1}\right)$. Let $E = \bigcup_{n \in \mathbb{N}} (E_n^+ \cup E_n^-)$. Claim: $\delta(0, E) = \alpha$.

Let $\delta > 0$, and suppose $\frac{1}{n+1} \leq \delta < \frac{1}{n}$ for $n \in \mathbb{N}$. Let $\delta = \frac{1}{n+1} + \varepsilon$. Then

$$\frac{m(E \cap (-\delta, \delta))}{2\delta} = \frac{1}{2\delta}\left[\frac{2\alpha}{n+1} + m\right]$$

$$= \frac{n+1}{2(\varepsilon n + \varepsilon + 1)}\left[\frac{2\alpha}{n+1} + m\right]$$

$$= \frac{\alpha}{\varepsilon n + \varepsilon + 1} + \frac{m(n+1)}{2(\varepsilon n + \varepsilon + 1)}$$

where $m = m\left(E \cap \left(\frac{1}{n+1}, \delta\right)\right) + m\left(E \cap \left(-\delta, \frac{1}{n+1}\right)\right)$.

Note $0 \leq m \leq \alpha\left(\frac{1}{n} - \frac{1}{n+1}\right)$ so $0 \cdot (n+1) \leq m(n+1) \leq \alpha\left(\frac{n+1}{n} - 1\right)$ and by the squeeze Theorem $\lim_{n\to\infty} m(n+1) = 0$. Further, since $0 \leq \varepsilon \leq \frac{1}{n} - \frac{1}{n+1}$, $0 \cdot n \leq \varepsilon n \leq 1 - \frac{n}{n+1}$, $\lim_{n\to\infty} \varepsilon n + \varepsilon + 1 = 1$ (again by squeeze).

combining all this we see that

$$\lim_{n\to\infty} \frac{m(E\cap(-\delta,\delta))}{2\delta} = \alpha$$

as desired.

c) Use the same construction as b). In $\mathbb{R}$, cubes and balls are both intervals. We can use the exact same arguement as in b), except with minor adjustments for the fact that the interval may not be symmetric about 0, bounding m using the smallest n possible for a given side length δ.

d.) Yes, this is possible. Allan decamp of Wesleyan University provides a construction using the Cantor Set.

(66) Since $\mathbb{Q}$ is countable, we may enumerate the rationals $q_1, q_2, q_3, \ldots$. Then, define

$$f(x) = \sum_{k \in S_x} \frac{1}{2^k}$$

where

$$S_x = \{ k \in \mathbb{N} : q_k \leq x \}$$

Clearly, $f(x)$ is monotone increasing. We claim that the discontinuity set is precisely $\mathbb{Q} \cap [0,1]$.

Let $q_n \in \mathbb{Q} \cap [0,1]$. Note that for any $x \in [0,1]$ s.t. $x < q_n$, $f(x) \leq f(q_n) - \frac{1}{2^n}$. Then, if f were continuous, we would have $\lim_{x \to q_n^-} f(x) = f(q_n)$. But this would imply $0 \leq -\frac{1}{2^n}$, a contradiction. So f is discontinuous at each $q_n \in \mathbb{Q} \cap [0,1]$.

Now, we show f is continuous at each $x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$. We want to show that $\lim_{y \to x} f(y) = f(x)$. Order the elements of S_x $k_1, k_2, \ldots$. Then $f(x) = \sum_{i=1}^{\infty} \frac{1}{2^{k_i}}$. Note that $\lim_{n \to \infty} \left(\sum_{i=n}^{\infty} \frac{1}{2^{k_i}} \right) = 0$, so for $\varepsilon > 0$, we can find n s.t. $\sum_{i=n}^{\infty} \frac{1}{2^{k_i}} < \varepsilon$. Let $q = \max(q_1, \ldots q_n)$. Then for $y \in (q, x)$, $|f(y) - f(x)| < \varepsilon$ so $\lim_{y \to x^-} f(y) = f(x)$.

The limit from above is similar. For $\tilde{x} > x$, we can order the elements of $S_{\tilde{x}} \setminus S_x$ $l_1, l_2, \ldots$ and $f(\tilde{x}) - f(x) = \sum_{i=1}^{\infty} \frac{1}{2^{l_i}}$. For $\varepsilon > 0$, we can find

n s.t. $\sum_{i=n}^{\infty} \frac{1}{2^{i}} < \varepsilon$. Then, letting $\hat{q} = \min(q_1, \dots, q_n)$, for $y \in (x, \hat{q})$, $|f(y) - f(x)| < \varepsilon$ and thus $\lim_{y \to x^+} f(y) = f(x)$ as desired.

(Inspired by p4sch's suggestion on Math StackExchange)