Math 105 HW 8

(83) Let $B = \prod_{i=1}^{n} [a_i, b_i]$. Suppose $(f_n) \to f$ a.e. on B. Then, $\forall \varepsilon > 0$, $\exists X$ s.t. $m(X^c) \le \varepsilon$ and $(f_n) \to f$ uniformly on X.

Proof:

Let

$$X(k, l) = \{x \in B : \forall n \ge k \text{ we have } |f_n(x) - f(x)| < 1/l\}$$

For a fixed $l \in \mathbb{N}$, since $f_n(x) \to f(x)$ a.e., $\bigcup_k X(k, l) \cup Z(l) = [a, b]$ for a zero set $Z(l)$.

Let $\varepsilon > 0$. Measure continuity implies $m(X(k,l)) \to b - a$ as $k \to \infty$ because $X(k, l) \to [a, b]$. So, we can pick $k_1 < k_2 < \dots$ s.t. for $X_l = X(k_l, l)$ $m(X_l^c) < \varepsilon / 2^l \Rightarrow m(X^c) < \varepsilon$ where $X = \bigcap_l X_l$, as $X^c = \bigcup_l X_l^c$.

Claim: $f_n \to f$ uniformly on X. Let $\delta > 0$, $1/l < \delta$. $\forall n \ge k_l$,

$x \in X \Rightarrow x \in X_l = X(k_l, l) \Rightarrow |f_n(x) - f(x)| < 1/l < \delta$

So $f_n \to f$ uniformly on X^c.

b) True. We can follow the same proof technique, since each $X(k,l)$ will have finite measure as it is a subset of the domain, which has finite measure.

c) We can use the "moving bump" example, where $f_n(x) = 1$ if $x \in (n, n+1)$ and 0 otherwise. The sequence converges pointwise to 0, but on $\mathbb{R} \setminus B$ for any finite subinterval $B \subseteq \mathbb{R}$, does not converge uniformly to 0.

d)

③ $\|T\|$ is the maximum singular value of T. To see why, we can write $T = U \Sigma V^T$, the SVD of T. Then $|\Sigma v|$ is maximized when $v = e_1$ (assuming Σ is arranged s.t. $\Sigma_{1,1}$ has the largest singular value). Since U, V^T are unitary, this implies that the largest singular value is also the maximum of $|Tv|$ when $|v| \leq 1$.

Optional: