Math 105 HW 9

9.6

If $(x,y) \neq (0,0)$, since f is a quotient of differentiable, non-zero functions near (x,y) we may simply compute

$$D_1 f(x,y) = y(x^2+y^2)^{-1} - 2x^2y(x^2+y^2)^{-2}$$
$$D_2 f(x,y) = x(x^2+y^2)^{-1} - 2xy^2(x^2+y^2)^{-2}$$

Now let $(x,y) = (0,0)$. We need to show that

$$\lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

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exist. We have

$$\lim_{h \to 0} \frac{1}{h}\left(\frac{h \cdot 0}{h^2+0^2} - 0\right) =$$

$$\lim_{h \to 0} 0 = 0$$

As desired (the proof for the second limit is similar).

It remains to show that f is not continuous @ $(0,0)$.

Note that

$$\lim_{x \to 0} f(x, x) = \lim_{x \to 0} \frac{x^2}{2x^2}$$

$$= \lim_{x \to 0} \frac{1}{2}$$

$$= \frac{1}{2} \neq f(0,0) = 0.$$

Thus, along the line $y = x$ the limit of f approaching the origin is not $f(0,0)$, so f is not continuous at $(0,0)$.

9.7

Let $x \in E$, and let $\varepsilon > 0$. Pick M_i s.t. $|D_i f_i| < M_i$ for each i, which is possible by boundedness of the partials. Let $M = \max_i M_i$, and let $r = \frac{\varepsilon}{nM}$.

Now, let $h = \sum h_j e_j$, $|h| < r$. Let $V_0 = 0$, $V_k = h_1 e_1 + \dots + h_k e_k$ for $1 \le k \le n$. Then,

$$f(x+h) - f(x) = \sum_{j=1}^{n} [f(x + v_j) - f(x + V_{j-1})].$$

Note that for each k, $|v_k| < r$ since $|v_k| < |h|$, so $x + v_k \in B(x, r)$. Since $v_k = v_{k-1} + h_k e_k$, by the MVT

$$f(x+v_k) - f(x+v_{k-1}) = h_k (D_k f)(x + v_{k-1} + \theta_k h_k e_k)$$

for some $\theta_k \in (0,1)$.

Then,

$$\begin{aligned}
|f(x+h) - f(x)| &= \left| \sum_{j=1}^{n} [f(x+v_j) - f(x+v_{j-1})] \right| \\
&\leq \sum_{j=1}^{n} |f(x+v_j) - f(x+v_{j-1})| \\
&= \sum_{j=1}^{n} |h_j (D_j f)(x + v_{j-1} + \theta_j h_j e_j)| \\
&\leq \sum_{j=1}^{n} |h_j| M \\
&\leq \sum_{j=1}^{n} \frac{\varepsilon}{n \cdot M} M \\
&= \sum_{j=1}^{n} \frac{\varepsilon}{n} \\
&= \varepsilon
\end{aligned}$$

This shows that $\lim_{h \to 0} f(x+h) = f(x)$ and that f is continous, as desired.

③ Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x) := d(x, E)$ where we define

$$d(x, E) = \inf \{ |x - y| : y \in E \}$$

Clearly, for $x \in E$, $f(x) = 0$. Now suppose $x \in \mathbb{R}^2 \setminus E$. Then since E is closed, $\mathbb{R}^2 \setminus E$ is open, so $\exists$ $\varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq \mathbb{R}^2 \setminus E$. Since $B(x, \varepsilon) \cap E = \emptyset$ we must have $d(x, E) \geq \varepsilon > 0$ so $f(x) \neq 0$, and $f^{-1}(0) = E$ as desired.

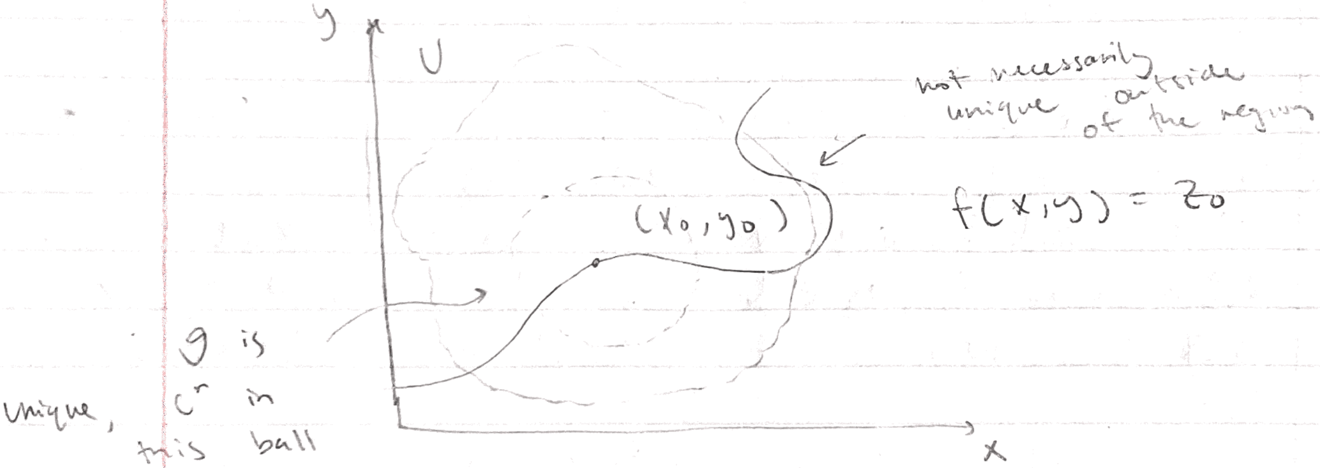
It remains to show that f is continuous. Let $x \in \mathbb{R}^2$. Let $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{2}$, and $x' \in B(x, \delta)$. WLOG suppose $d(x, E) \leq d(x', E)$. $\exists$ $y \in E$ s.t. $d(x, E) + \frac{\varepsilon}{2} > d(x, y)$. Note that $d(x', y) \geq d(x', E)$. Thus

$$\begin{aligned}
|f(x) - f(x')| &= |d(x, E) - d(x', E)| \\
&= d(x', E) - d(x, E) \\
&\leq d(x', y) - d(x, E) \\
&\leq d(x', y) - d(x, y) + \tfrac{\varepsilon}{2} \\
&\leq \big| |x' - y| - |x - y| \big| + \tfrac{\varepsilon}{2} \\
&\leq |x' - x| + \tfrac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

by the reverse triangle inequality. This shows continuity as desired.

④ Implicit Function Theorem for $n=m=1$:

Let $f: U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^2$ is open. If f is C^r for $1 \leq r \leq \infty$ near a point (x_0, y_0), then the z_0-locus of f (the set of points where $f(x,y) = z_0$) is the graph of a unique function $y = g(x)$, where g is C^r.



Basically, this theorem says that $f(x,y) = z_0$ is a function defined implicitly in terms of x and y in a small enough neighborhood of a point. The condition that f is C^r is necessary because it ensures that f is well-behaved enough near (x_0, y_0). For instance, it is possible that the entire graph of $f(x,y) = z_0$ is, say, a circle, which cannot be written as a function. But if you restrict to a small enough region it can be.