Lecture 10: 02/17/22

Prop: if f, g simple fn then

$$\int f+g = \int f + \int g$$

Recall: If f simple,

$$f(x) = \sum_{i=1}^{N} c_i 1_{E_i}(x)$$

$$\int f = \sum_{i=1}^{N} c_i m(E_i)$$

Pf: $f = \sum_{i=1}^{N} c_i 1_{E_i}(x)$ $c_i > 0$, E_i disjoint

$$g = \sum_{i=1}^{M} d_i 1_{F_i}(x)$$

Let $E_0 = \mathbb{R}^n \setminus E_1 \cup \dots \cup E_N$ $c_0 = 0$

$F_0 = \mathbb{R}^n \setminus F_1 \cup \dots \cup F_M$ $d_0 = 0$

Then

$$\mathbb{R}^n = \bigcup_{i=0}^{N} \bigcup_{j=0}^{M} E_i \cap F_j$$

$$f+g = \sum_{i,j=0}^{N,M} (c_i + d_j) 1_{E_i \cap F_j}$$

$$\int f+g = \sum_{i=0}^{N} \sum_{j=0}^{M} (c_i + d_j) m(E_i \cap F_j)$$

$$= \sum_{i,j} c_i m(E_i \cap F_j) + \sum_{i,j} d_j m(E_i \cap F_j)$$

$$= \sum_{i} c_i m(E_i) + \sum_{j} d_j m(E_i \cap F_j)$$

$$= \int f + \int g$$

Integration of non-neg meas fun

Def: Let $f \geq 0$ be measurable, $\Omega \to [0, \infty)$ $(\Omega \subseteq \mathbb{R}^n)$

$$\int f = \sup\{ \int s \mid s \text{ simple}, s \geq 0, s \leq f \}$$

Prop: If $f, g : \Omega \to [0, \infty]$

① $\int f \geq 0$, + $\int f = 0$ iff $f(x) = 0$ a.e

② $\forall c > 0$ $\int c \cdot f = c \cdot \int f$

Pf: ① If $f(x) = 0$ a.e., for any simple s, $0 \leq s(x) \leq f(x)$

$s = \sum c_i 1_{E_i}(x)$, E_i (null) $\subset Z = \{x \mid f(x) > 0\}$

$\Rightarrow \int s = \sum c_i m(E_i) = \sum c_i \cdot 0 = 0$

Prop: ③ $f \leq g \Rightarrow \int f \leq \int g$

④ if $f = g$ a.e then $\int f = \int g$

Pf: Let $Z = \{x \mid f(x) \neq g(x)\}$, $Z^c = \Omega \setminus Z$

$\forall s$ simple

$\int s = \int s \cdot 1_{Z_c} \Rightarrow \int f = \int f \cdot 1_{Z_c} = \int g \cdot 1_{Z_c} = \int g$

Thm: Given meas fun

$f_n : \Omega \to [0, \infty]$

$0 \leq f_1(x) \leq f_2(x) \leq \dots$

Then

① $0 \leq \int f_1 \leq \int f_2 \leq \dots$

② $\int \sup f_n(x) = \sup \int f_n$

Pf: $\sup f_n(x) = f(x) \geq f_n(x) \quad \forall n$

$\leadsto \int f \geq \int f_n \quad \forall n$

$\leadsto \int f \geq \sup \int f_n$

WTS

$\int f \leq \sup_n \int f_n$

$\Leftarrow \forall s$ simple, $0 \leq s \leq f$

$\int s \leq \sup_n \int f_n$

$\Leftarrow \forall\, 1 > \varepsilon > 0, \quad \forall s$ simple

$(1-\varepsilon) \int s \leq \sup_n \int f_n$

Define

$E_n = \{ x \in \Omega \mid f_n(x) \geq (1-\varepsilon) s(x) \}$

Then

$\subset E_n \subset E_{n+1} \subset \cdots$

$\bigcup E_n = \Omega$

$\int_{E_n} (1-\varepsilon) s_n \leq \int_{E_n} f_n \leq \int_\Omega f_n \leq \sup_n \int_\Omega f_n$

$\lim_{n\to\infty} \int_{E_n} (1-\varepsilon) s = \int_\Omega (1-\varepsilon) s$

b/c $s = \sum c_i 1_{F_i}(x)$

$\int_{E_n} s = \sum c_i \cdot m(F_i \cap E_n)$

$\lim \int_{E_n} s = \sum c_i m(F_i) = \int_\Omega s$

This shows the result.

Prop: If $f, g : \Omega \to [0, \infty]$ meas
then $\int f+g = \int f + \int g$

Pf: Let s_n be a seq of simple fn $s_n \nearrow f$ $t_n \nearrow g$

then, by MCT $\lim \int s_n = \int f$, $\lim \int t_n = \int g$
$\lim \int s_n + t_n = \int f + g$. $\because$ $\int t_n + s_n = \int s_n + \int t_n$
$\therefore$ $\lim \int t_n + s_n = \lim (\int s_n) + \lim (\int t_n)$

Cor: If $g_1, g_2, \ldots$ are non-neg meas - fun then

$$\int_\Omega \sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} \int_\Omega g_n(x)$$

Pf: Let $f_n = \sum_{m=1}^{n} g_m(x)$
Then

$$\begin{aligned} \int \sup f_n &= \sup \int f_n \\ &= \sup_N \int \sum_{n=1}^{N} g_n \\ &= \sup_N \sum_{n=1}^{N} \int g_n \end{aligned}$$

Prop: If $f : \Omega \to [0, \infty]$ meas, $\int f < \infty$
then $f(x)$ is finite a.e.

Cor: (Borel-Cantelli) If $\Omega_1, \Omega_2 \ldots$ are measurable sets s.t. $\sum_{i=1}^{\infty} m(\Omega_i) < \infty$

Then $\{x \mid x$ belongs to ∞ many $\Omega_n\}$ is a null set pf: $m(\Omega_i) = \int_{\Omega_i} 1 = \int_{\Omega} 1_{\Omega_i}$

$\sum_{1}^{\infty} m(\Omega_i) = \sum_{1}^{\infty} \int_{\Omega} 1_{\Omega_i} = \int_{\Omega} \underbrace{\sum_{i=1}^{\infty} 1_{\Omega_i}(x)}_{f(x)} < \infty$

$\therefore f^{-1}(\infty)$ is a null set.