Lecture 2 - 01/20/22

Lemma: Outer measure has the following 4 properties:

(V) $m^*(\emptyset) = 0$ (empty set)

- no box needed

(Vi) $0 \le m^*(\Omega) \le +\infty$ for every measurable set (positivity)

- inf over non-negative numbers

(Vii) $A \subseteq B \subseteq \mathbb{R}^n \Rightarrow m^*(A) \le m^*(B)$ (monotonicity)

- any open cover $\{B_i\}$ of B is also an open cover of A
- if $M, N \subseteq \mathbb{R}$, $M \supset N$ $\inf M \le \inf N$

(Viii) $(A_j)_{j\in J}$ finite, $m^*(\bigcup_{j\in J} A_j) \le \sum_{j\in J} m^*(A_j)$ (finite-subadditivity)

- induction, WTS $m^*(A \cup B) \le m^*(B) + m^*(A)$
- show $m^*(A) + m^*(B) \ge$ (total area of some covering A + covering of B) $- \varepsilon$

  then as $\varepsilon \to 0$ $m^*(A) + m^*(B) \ge m^*(A \cup B)$
- b/c $\forall \varepsilon > 0$, $\exists \{B_i\}$ s.t. $\sum_i |B_i| \le m^*(A) + \varepsilon/2$ and similar for B. Then take union of 2 countable covers to get cover of $A \cup B$

(X) $m^*(\bigcup_{j=1}^{\infty} A_j) \le \sum m^*(A_j)$ (countable sub-additivity)

- show $\forall \varepsilon > 0 \; \exists \{B_i^{(j)}\}$ for A_j s.t.

$$m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_j m^*(A_j) + \varepsilon$$
$$= \sum_{j=1}^{\infty} \left(m^*(A_j) + \frac{\varepsilon}{2^j}\right)$$

- can find open cover $\{B_i^{(j)}\}$ for A_j s.t.

$$m^*(A_j) + \frac{\varepsilon}{2^j} \ge \sum_{i=1}^{\infty} |B_i^{(j)}|$$

and $\sum^{\infty} \left(\sum_{i=1}^{\infty} |B_i^{(j)}| \right) \ge m^*\left(\bigcup_j^{\infty} A_j\right)$

(xiii) $m^*(x+\Omega) = m^*(\Omega)$ (translation invariance)

- $\{B_i\}$ covers Ω $\Rightarrow$
  $\{B_i + x\}$ covers $\Omega + x$

<u>Prop:</u> For any closed box
$B = \prod_{i=1}^{n} [a_i, b_i] = \{(x_1, \dots x_n) \in \mathbb{R}^n : x_i \in [a_i, b_i] \ \forall\ 1 \le i \le n\}$

we have
$$m^*(B) = \prod_{i=1}^{n} (b_i - a_i)$$

<u>Proof:</u> · compact set in $\mathbb{R}^n$ $\Leftrightarrow$ closed + bounded

· Riemann integral

(1D) $\text{vol}([a,b]) = b - a = \int_a^b 1 \, dx = \int_{\mathbb{R}} 1_{[a,b]} \, dx$

$$1_{[a,b]}(x) = \begin{cases} 1 & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

(n D) $\text{vol}(\underbrace{[a_1,b_1] \times \dots \times [a_n,b_n]}_{B}) = \int_{\mathbb{R}^n} 1_B(x) \, dx$

(same for open boxes)

- $n = 1$ case:
$B = [a,b]$ compact $\Rightarrow$ any open cover can be reduced to a finite subcover. Let $\{B_i\}_{i=1}^{N}$ be a finite open cover of B. Then, WTS
$$\sum_{i=1}^{N} |B_i| \ge \text{vol}(B)$$

Let $f_i(x) = 1_{B_i}(x)$. Then

$$\sum_{i=1}^{N} |B_i| = \sum_{i=1}^{N} \left(\int_{\mathbb{R}} f_i \, dx \right)$$

$$= \int_{\mathbb{R}} \underbrace{\sum_{i=1}^{N} f_i}_{f(x)} \, dx$$

Claim: $f(x) \geq 1_B(x)$ indeed, $B \subset \bigcup_{i=1}^{N} B_i$

thus $1_B \leq \sum 1_{B_i}$

$$\leadsto \sum_{i=1}^{N} |B_i| \geq \int 1_B \, dx = vol(B)$$

can't be greater than $vol(B)$ b/c can take arbitrarily small open interval around $[a,b]$.

. $n=2$ case:

WTS $\forall \{B_i\}_{i=1}^{N}$ of B $\sum_{i=1}^{N} |B_i| > |B|$

again $|B_i| = \int_{\mathbb{R}^2} 1_{B_i}(x_1, x_2) \, dx_1 \, dx_2 = \int_{\mathbb{R}} w_i \cdot 1_{B_{i,1}}(x_1) \, dx_1$

going to integrate along x_2

$$\sum_{i=1}^{N} \int 1_{B_i}(x) \, dx_1 \, dx_2 =$$

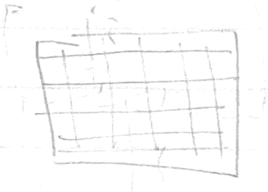
$$\int_{\mathbb{R}^2} \sum_{i=1}^{N} 1_{B_i}(x) \, dx_1 \, dx_2 = \int_{\mathbb{R}} \Big(\overbrace{\int \sum_i 1_{B_i}(x_1, x_2) \, dx_2}^{f(x_1)} \Big) \, dx_1$$

Claim: $f(x_1) \geq 1_{[a_1, b_1]}(x_1) \underbrace{|b_2 - a_2|}_{\text{height of } B}$

follows by induction hypothesis ($n=1$ case) applied to line w/ given x_1

$$\geq \int_{\mathbb{R}} 1_{[a_1, b_1]} (b_2 - a_2) \, dx_1 = (b_2 - a_2)(b_1 - a_1)$$
$$= vol(B)$$

Pugh:



- divide B into grids of smaller boxes s.t. each small box is contained in some B_i
- then $Vol(B) = \sum$ vol of grid small boxes
  $\leq \sum$ vol of open cover B_i

Corr: Outer measure of any box (open, closed, half open/closed like $(a_1, b_1) \times (a_2, b_2]$)
$=$ vol(box)

① $m^*(\mathbb{N}) = 0$ (countable sub additivity)

② $m^*(\mathbb{Q}) = 0$ (similar)

③ $m^*(\mathbb{R}) = \infty$

$m^*((-R, R)) = 2R$

$m^*(\mathbb{R}) \geq 2R \quad \forall R > 0$

take lim as $R \to \infty$

Tao 7.3'

idea:

① construct weird subset $E \subset [0,1]$

② $[-1, 2] \supset \bigsqcup_{q \in [-1,1] \cap \mathbb{Q}} q + E \supset [0,1]$

$\rightsquigarrow$ additivity fails:

$m^*\left(\bigsqcup_{q \in [-1,1] \cap \mathbb{Q}} q + E\right) = \sum_{q \in [-1,1] \cap \mathbb{Q}} m^*(q + E)$

$= \sum_{q \in} m^*(E) = 0 \,?\quad +\infty\,?$

$$m^*([0,1]) \leq m^*\left(\bigsqcup_q q+E\right) \leq m^*([-1,2]) = 3$$
$$\overset{\shortparallel}{1}$$

Exercise: Half-spaces are measurable

- $\mathbb{R}_1$ case: $\{x \in \mathbb{R}, \ x > 0\} = H$