Counterexample for m^*

Lecture 3 - 01/25/22

- A coset of $\mathbb{Q}$ if $A = x + \mathbb{Q}$ for some $x \in \mathbb{R}$
- $x + \mathbb{Q}$ and $y + \mathbb{Q}$ are either disjoint or entirely the same
  - cosets have non-empty intersection w/ $[0,1]$
- for each coset, pick a representative x_A s.t. $x_A \in [0,1]$. Construct $E = \{x_A\}$. $X = \bigcup_{q \in \mathbb{Q} \cap [-1,1]} (q + E)$
- due to construction of X, $[0,1] \subset X \subset [-1,2]$
- $x \in X$ can be written $x = q + e$ where $q \in \mathbb{Q} \cap [-1,1]$ and $e \in E$ $\Rightarrow x \in [-1,2]$ so $m^*(X) \leq 3$ by monotonicity
- for $x \in [0,1]$, $x \in x + \mathbb{Q}$, so $\exists\, y \in E$ s.t. $y + q = x$ where $q \in \mathbb{Q}$. Since $|x - y| \leq 1$, $q \in [-1,1] \Rightarrow x \in X$, thus $m^*(X) \geq 1$
- Since $X = \bigcup_{q \in \mathbb{Q} \cap [-1,1]} (q + E)$ and $\mathbb{Q} \cap [-1,1]$ countable, $m^*(X) = \sum m^*(q + E)$, $m^*(q+E) = m^*(E)$
  $\Rightarrow$ RHS is either 0 or ∞ $\Rightarrow \epsilon$

Defn: E is measurable iff

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E) \quad \forall A \subseteq \mathbb{R}^n$$

Lemma: (half spaces measurable) i.e. $\{(x_1, \ldots x_n) \mid x_n > 0\}$ is measurable in $\mathbb{R}^n$

Pf ($n=1$): WTS $\forall A \subset \mathbb{R}$

$m^*(A) = m^*(A_+) + m^*(A_-)$

where $A_+ = A \cap (0, \infty)$ $A_- = A \cap (-\infty, 0]$

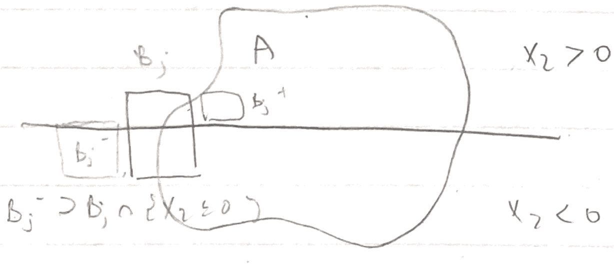
① By sub-additivity
$m^*(A) \le m^*(A_+) + m^*(A_-)$

② To show $\ge$ direction, show $\forall \varepsilon > 0$
$m^*(A) + \varepsilon \ge m^*(A_+) + m^*(A_-)$
Consider open cover of A by $\{B_j\}$
s.t. $\sum |B_j| \le m^*(A) + \varepsilon/2$

Define $B_j^+ = B_j \cap (0, \infty)$, $B_j^- = B_j \cap (-\infty, \frac{\varepsilon}{2^{j+1}})$
Then $B_j = B_j^- \cup B_j^+$ and $|B_j| + \frac{\varepsilon}{2^{j+1}} \ge |B_j^+| + |B_j^-| \ge |B_j|$
$\bigcup B_j^+ \supset A_+$, $\bigcup B_j^- \supset A_-$

Thus
$$m^*(A_+) + m^*(A_-) \le \sum |B_j^+| + \sum |B_j^-| \le \sum_{j=1}^{\infty} \left(|B_j| + \frac{\varepsilon}{2^{j+1}}\right)$$
$$\le \left(\sum |B_j|\right) + \frac{\varepsilon}{2}$$
$$\le m^*(A) + \varepsilon.$$

(Try $n = 2$)



$A_+ = A \cap \{x_2 > 0\}$
$A_- = A \cap \{x_2 \le 0\}$

need to show $m^*(A) + \varepsilon \ge m^*(A_+) + m^*(A_-)$

can do similar thing as $n = 1$:
① get $\{B_j\}$ cover of A w/ $m^*(A) + \frac{\varepsilon}{2} \ge \sum |B_j|$
② $B_j^+ = B_j \cap \{x_2 > 0\}$ $B_j^- = B_j \cap \{x_2 < \varepsilon_j\}$
s.t. $|B_j^-| + |B_j^+| \le |B_j| + \frac{\varepsilon}{2^{j+1}}$

More systematic approach:

- for A = open box in $\mathbb{R}^n$, prove $m^*(A) = m^*(A_+) + m^*(A_-)$
- for general A, $\forall\, \varepsilon > 0$, find $\{B_j\}$ cover of A s.t. $m^*(A) + \varepsilon \geq \sum |B_j|$

define $B_j^+ = B_j \cap \{x_n > 0\}$, $B_j^- = \underbrace{B_j \cap \{x_n \leq 0\}}_{\text{may not be open}}$

- $\sum_j |B_j^+| + |B_j^-| = \sum |B_j^+| + \sum |B_j^-| = \sum |B_j|$
- Since $A_+ \subset \bigcup B_j^+$, $m^*(A_+) \leq \sum m^*(B_j^+)$ $= \sum |B_j^+|$
- Since $A_- \subset \bigcup B_j^-$, $m^*(A_-) \leq \sum m^*(B_j^-)$ $= \sum |B_j^-|$
- Thus $m^*(A_+) + m^*(A_-) \leq m^*(A) + \varepsilon$

Lemma (prop of measurable set):

a) if $E \subset \mathbb{R}^n$ meas, then E^c is meas.
(true by def)

b) translation invariance, if E meas then $\forall\, x \in \mathbb{R}^n$, $x + E$ is measurable

(pf: $\forall\, A \subseteq \mathbb{R}^n$, $m^*(A) \stackrel{?}{=} m^*(A \cap (x+E)) + m^*(A \cap (x+E)^c)$

$\Leftrightarrow$ $m^*(A-x) = m^*((A-x) \cap E) + m^*((A-x) \cap E^c)$

c) If E_1, E_2 measurable, then $E_1 \cap E_2$, are too

(pf: let

$A_{++} = A \cap E_1 \cap E_2$ $\qquad$ $A_{+-} = A \cap E_1 \cap E_2^c$

$A_{-+} = A \cap E_1^c \cap E_2$ $\qquad$ $A_{--} = A \cap E_1^c \cap E_2^c$

$A = A_{++} \sqcup A_{+-} \sqcup A_{-+} \sqcup A_{--}$

$m^*(A) = m^*(A_{++}) + m^*(A_{+-} \cup A_{-+} \cup A_{--})$

$m^*(A) = m^*(A_{+-} \cup A_{++}) + m^*(A_{-+} \cup A_{--})$

$\quad\quad\quad\quad\quad \parallel \quad\quad\quad\quad\quad\quad \parallel$

$\quad\quad\quad\quad A \cap E_1 \quad\quad\quad\quad A \cap E_1^c$

$$\left.\begin{aligned} m^*(A_{+-} \cup A_{++}) &= m^*(A_{+-}) + m^*(A_{++}) \\ m^*(A_{-+} \cup A_{--}) &= m^*(A_{--}) + m^*(A_{-+}) \end{aligned}\right\} E_2 \text{ measurable}$$

(need some more args)

$m^*(A_{+-} \cup A_{-+} \cup A_{--}) = m^*(A_{+-}) + m^*(A_{-+}) + m^*(A_{--})$

d) (Boolean alg) finite intersection / union preserve measurability (using induction)

e). every box (open or closed) is measurable

f) if $m^*(E)=0$, E is measurable

(Pf: $\forall A \subset \mathbb{R}^n$, need

$m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E)$

w/ $m^*(A \cap E) \leq m^*(E) = 0 \Rightarrow m^*(A \cap E) = 0$

$m^*(A) \geq m^*(A \setminus E)$ by monotonicity

<u>Lemma</u> (finite additivity): If $E_1, \dots, E_n$ are disjoint, measurable, then $\forall A \subset \mathbb{R}^n$

$m^*(A \cap E) = \sum m^*(A \cap E_i)$ $\quad (E = \bigcup_{i=1}^{n} E_i)$