Lecture 4 - 01/27/22

Lemma: A, B measurable, $\Rightarrow$ so is $B \setminus A$, and

$$m^*(B \setminus A) = m^*(B) - m^*(A)$$

Pf: $B \setminus A = A^c \cap B$ which is measurable. Eqn from measurability of A applied to B.

Prop: (Countable Additivity): Let $\{E_j\}_{j=1}^\infty$ countable collection of disjoint measurable sets. WTS

- $E = \bigcup_{j=1}^\infty E_j$ measurable
- $m^*(E) = \sum_{j=1}^\infty m^*(E_j)$

Pf: WTS $\forall A \subset \mathbb{R}^n$
$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$

Define $F_N = \bigsqcup_{j=1}^N E_j$

- F_N measurable (finite union)
- $m^*(F_N) = \sum_{j=1}^N m^*(E_j)$ (additivity)

If we replace E by F_N, $E \supset F_N$, $E^c \subset F_N^c$

- $m^*(A \cap E) \geq m^*(A \cap F_N)$
- $m^*(A \cap E^c) \leq m^*(A \cap F_N^c)$

$\leq$ direction follows from sub-additivity

$$m^*(A \cap E) \leq \sum_{j=1}^\infty m^*(A \cap E_j)$$ by countable sub additivity

$$= \sup_N \left(\sum_{j=1}^N m^*(A \cap E_j) \right)$$

$$= \sup_N m^*(A \cap F_N)$$ (finite additivity)

Thus,

$$m^*(A\cap E) + m^*(A\cap E^c) \le \left[\sup_N m^*(A\cap E_N)\right] + m^*(A\cap E^c)$$
$$\le \sup_N \left(m^*(A\cap F_N) + m^*(A\cap E^c)\right)$$
$$\le \sup_N \left(m^*(A\cap F_N) + m^*(A\cap F_N^c)\right)$$
$$= \sup_N \left[m^*(A)\right] = m^*(A)$$

$m^*(E) \le \sum_j m^*(E_j)$ subadditivity

$m^*(E) \ge m^*(F_N) = \sum_{j=1}^N m^*(E_j)$

monotonicity

then take sup over N

Prop: Set of measurable sets forms a σ-algebra i.e. given any countable collection Ω_j of measurable sets, $\bigcap_{j=1} \Omega_j$ & $\bigcup_{j=1} \Omega_j$ are measurable

Pf: Consider $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. Let $\Omega_N = \bigcup_{j=1}^N \Omega_j$, $E_N = \Omega_N \setminus \Omega_{N-1}$. Then $\{\Omega_N\}$ meas, $\{E_N\}$ meas. Ω meas by previous prop ($\Omega = \bigcup_{j=1}^{\infty} E_j$)
$\bigcap_{j=1}^{\infty} \Omega_j = \left(\bigcup_{j=1}^{\infty} \Omega_j^c\right)^c$ is measurable b/c complements preserve meas.

Lemma: All open sets in $\mathbb{R}^n$ can be written as a countable union of open boxes

Topology

- open ball $B(x,r) = \{y \in X \mid d(y,x) < r\}$
  $x \in X$, $r > 0$ real
- open sets in X generated from open balls, by taking finite intersections & arbitrary unions

- equivalently $\mathcal{U} \subset X$ open iff $\forall x \in \mathcal{U}$ $\exists r > 0$ s.t. $B(x, r) \subset \mathcal{U}$

- topology for product space
  - If X, Y are top. spaces, $X \times Y$ can be endowed w/ product topology i.e. $W \subset X \times Y$ open if $\forall (x, y) \in W$ $\exists\, \mathcal{U} \subset X,\ V \subset Y$ open s.t. $(x, y) \in \mathcal{U} \times V \subset W$

- topology on $\mathbb{R}^2$
  - can be generated by open balls using Euclidean metric on $\mathbb{R}^2$
  - can be generated by open boxes

Pf of lemma: Consider the set of "rational boxes". A box $\prod_{i=1}^{n} (a_i, b_i)$ is rational if $a_1, b_1 \ldots a_n, b_n \in \mathbb{Q}$

The collection of {rational boxes} $\subset \mathbb{Q}^{2n}$ so is countable.

Suffice to show that every open set in $\mathbb{R}^n$ is a union of rational boxes, i.e. If $\mathcal{U}$ open $x \in \mathcal{U}$, want to find a rational open box B s.t. $x \in B \subset \mathcal{U}$. Since $\mathcal{U}$ open $\exists r > 0$ s.t. $x \in B(x, r) \subset \mathcal{U}$

Claim: $\exists$ rational box B s.t. $x \in B \subset B(x, r)$

Prop: Open sets in $\mathbb{R}^n$ measurable

∵ open set is a countable union of open boxes

Alternative defn of measurable set

Defn: $E \subset \mathbb{R}^n$ is measurable if $\forall \varepsilon > 0$ $\exists$ open $\mathcal{U}$ s.t. $\mathcal{U} \supset E$ $m^*(\mathcal{U} \setminus E) < \varepsilon$