Lecture 5 - 02/01/22

Abstract Measure Theory

- S set 2^S = set of all subsets in S
- Defn (σ-algebra): A subset $M_S \subset 2^S$ is called a σ-alg if
  1. $\phi \in M_S$
  2. If $A_1, A_2, \ldots \in M_S$ countable, then $\bigcup_{n=1}^{\infty} A_n \in M_S$
  3. $A \in M_S \Rightarrow A^c \in M_S$

(S, M_S) is called "measurable space".

- Defn: A measure ω on (S, M_S) is a function $\omega : M_S \to [0, +\infty]$ s.t.
  1. $\omega(\phi) = 0$
  2. if $A_1, A_2 \ldots \in M_S$ disjoint, then $\omega(\bigcup A_n) = \sum \omega(A_n)$

(S, M_S, ω) is a measurable space

Defn: (measurable function)
Given (X, M_X) & (Y, M_Y), we say a measurable map/function from X to Y
$f: X \to Y$ s.t. $\forall E \in M_Y$, we have $f^{-1}(E) \in M_X$

- If $(X, M_X) \xrightarrow{f} (Y, M_Y) \xrightarrow{g} (Z, M_Z)$

$g \circ f$ is measurable

If S is a topological space, i.e.
$\tau_S \subset 2^S$ collection of open subsets
- $\phi \in \tau_S$
- If $U_1, \dots U_n \in \tau_S$
  $U_1 \cap \dots \cap U_n \in \tau_S$
- arbitrary union of open is open.

There is a <u>minimal</u> σ-alg, containing τ_S. $\langle \tau_S \rangle$: <u>Borel σ-algebra</u>

<u>Q:</u> on $\mathbb{R}^n$ is Borel measurable $\Leftrightarrow$ Lebesgue measurable?

<u>Defn:</u> if S is a top. space, we say
- if $U_1, U_2, \dots$ countable collection of open sets, then
  $\bigcap_{n=1}^{\infty} U_n$ is called a <u>G_δ-set</u>
- if $F_1, F_2, \dots$ are closed, then
  $\bigcup_{n=1}^{\infty} F_n$ is called a <u>F_σ-set</u>

<u>Defn:</u> Let S be a set. an <u>outer measure</u> w on S is a function $w: 2^S \longrightarrow [0, \infty]$ s.t.

1. $w(\phi) = 0$
2. if $A \subset B$, $w(A) \le w(B)$
3. Countable sub-additivity - if $A_1, A_2 \dots$ $w(\bigcup_n A_n) \le \sum_{n=1}^{\infty} w(A_n)$

Construction: given w, define $M_S \subset 2^S$.
$E \in M_S$ if $\forall X \subset S$, $w(X) = w(X \cap E) + w(X \cap E^c)$

Thm: M_S is a σ-alg + w on M_S satisfies countable additivity.
(S, M_S, w) is a measure space

Defn: a subset $E \subset S$ w/ $w(E) = 0$ is called a "zero set" or "null set"

Lemma: ② $\forall A \subset S$, $w(A \cup E) = w(A)$
① If E null, if $E' \subset E$, E' is null

Pf: ① by monotonicity
② $w(A \cup E) = w(A) + w((A \cup E) \cap A^c)$
$= w(A) + w(E \cap A^c)$
$= w(A)$

Lemma: ③ $\forall A \subset S$, $w(A \cap E^c) = w(A)$
④ $w(E) = 0 \Rightarrow E$ measurable

Pf: ④ $\forall A \subset S$, WTS $w(A) = w(A \cap E) + w(A \cap E^c)$

But RHS simplifies to $w(A)$

Lemma: ⑤ If Z null set, then F is measurable $\Leftrightarrow F \cup Z$ is meas

Pf: ($\Rightarrow$) WTS $\forall A \subset S$

$$m(A) = m(A \cap (F \cup Z)) + m(A \cap (F \cup Z)^c)$$
$$= m((A \cap F) \cup (A \cap Z)) + m(A \cap (F^c \cap Z^c))$$
$$= m(A \cap F) + m(A \cap F^c)$$
$$= m(A) \quad (F \text{ meas})$$

Slogan: adding / subtracting null set does not effect measurability

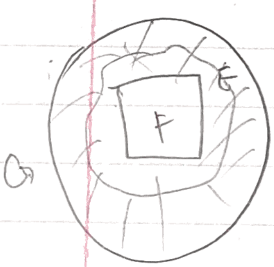
Example: $\{0\} \times \mathbb{R} \subset \mathbb{R}^2$ has measure 0

Pf: $\forall \varepsilon > 0$, $\exists$ open box cover of $\{0\} \times \mathbb{R}$ s.t. $\sum |B_j| < \varepsilon$

$$B_j = \left(-\frac{\varepsilon}{2^{n+2}}, \frac{\varepsilon}{2^{n+2}}\right) \times (-2^n, 2^n)$$

$$|B_n| = \frac{\varepsilon}{2^n}$$

Thm ($\mathbb{R}^n$, Lebesgue meas): $\forall E \subset \mathbb{R}^n$
E is Lebesgue measurable $\iff \exists$ F_σ-set and G_δ-set s.t. $F \subset E \subset G$ and $m(G \setminus F) = 0$



Pf: Assume E is bounded.
($\Rightarrow$) assume E Lebesgue measurable, and $E \subset R \leftarrow (-a, a)^n$ big box

$\forall n$, $\exists$ box covering $\{B_j^{(n)}\}$ of E s.t. $\sum |B_j^{(n)}| < m(E) + \frac{1}{n}$, let $U_n = \bigcup_j B_j^{(n)}$

Similarly, $\exists$ $V_n \supset \mathbb{R}\setminus E$, s.t. V_n open, $V_n \subset \mathbb{R}$ and $m(\mathbb{R}\setminus E) \leq m(V_n) \leq m(\mathbb{R}\setminus E) + \frac{1}{n}$

Define $G = \bigcap_{n=1}^{\infty} U_n$

$F_n = \mathbb{R}\setminus V_n$ closed, $F_n \subset E$

① $m(E) \leq m(U_n) \leq m(E) + \frac{1}{n}$

② $m(\mathbb{R}\setminus E) \leq m(V_n) \leq m(\mathbb{R}\setminus E) + \frac{1}{n}$

Claim: ② $\Leftrightarrow$ ②'

②' $m(F_n) \leq m(E) \leq m(F_n) + \frac{1}{n}$

$m(G/F) \leq m(U_n \setminus F_n) \leq \frac{2}{n}$ $\forall n$ $\square$