Lecture 7 - 02/08/22

Thm: $m(E) = 0 \iff$ almost every slice E_x has measure 0, i.e. $Z = \{x \mid m(E_x) > 0\} \Rightarrow m(Z) = 0$ $(Z \in \mathbb{R}^k)$

Last time: $\Leftarrow$

1. assume $Z = \emptyset$, E bdd
2. approx E from inside w/ compact subset K
3. cover K by finitely many open boxes
4. disjointize

$m(K) \le \sum m(U_i) \times m(V_i) < \varepsilon$

$\Rightarrow$

Lemma: $\forall$ W open, $\forall \alpha > 0$ let $X_\alpha = \{x \in \mathbb{R}^k \mid m(W_x) > \alpha\}$, $W \subseteq \mathbb{R}^k \times \mathbb{R}^n$.
then $m_{k+n}(W) \ge m_k(X_\alpha) \cdot \alpha$

Pf: 1. $\forall x \in X_\alpha$ get a compact set $K_x \subseteq W_x$, $m(K_x) > \alpha$. Let $U(x), V(x)$ open, $\{x\} \times K_x \subseteq U(x) \times V(x) \subseteq W$
$\mathbb{R}^k$ $\mathbb{R}^n$

Thus X_α is open, $U(x) \subseteq X_\alpha$, as $\forall$ $x' \in U(x)$, $W_{x'} \supset V(x) \supset K_x$.
$m(W_{x'}) \ge m(K_x) > \alpha$

2. $\forall K' \subset X_\alpha$ cpt $X_\alpha \subset \bigcup_{x \in X_\alpha} U(x)$
$\Rightarrow$ get finite subcover for K'

$K' = \mathcal{U}(x_1) \cup \mathcal{U}(x_2) \dots \cup \mathcal{U}(x_n)$, Let
$\mathcal{U}_1 = \mathcal{U}(x_1)$, $\mathcal{U}_2 = \mathcal{U}(x_2) \setminus \mathcal{U}(x_1) \dots$ $\mathcal{U}_i$ disjoint.

$$m(W) \geq m\left(\bigcup_{i=1}^{N} \mathcal{U}_i \times V(x_i)\right) \geq \sum m(\mathcal{U}_i) \times \alpha$$
$$\geq m(K') \times \alpha$$

Lebesgue Integral
Let $f: \mathbb{R} \to [0, \infty)$

Defn: Undergraph $\mathcal{U}(f) = \{(x,y) \mid 0 \leq y < f(x)\}$
f is measurable if $\mathcal{U}(f)$ is a measurable subset.

$\int f = m(\mathcal{U}(f))$ (possibly $= +\infty$). if $\int f < \infty$, f is integrable

a.e. almost everywhere = "up to a measure 0 set"

Thm: Let $f_n: \mathbb{R} \to [0, \infty]$ be a seq of meas. func + $f_n \nearrow f$ a.e. as $n \to \infty$. Then $\int f_n \to \int f$

Pf: $f_n \nearrow f \Rightarrow \mathcal{U}(f_n) \nearrow \mathcal{U}(f) \Rightarrow$
$m(\mathcal{U}(f_n)) \nearrow m(\mathcal{U}(f))$

Def. Completed undergraph $\hat{\mathcal{U}}(f) = \{(x,y) \mid 0 \leq y \leq f(x)\}$

Prop: $\mathcal{U}(f)$ meas $\Leftrightarrow$ $\hat{\mathcal{U}}(f)$ meas
$m(\mathcal{U}(f)) = m(\hat{\mathcal{U}}(f))$

Fact: If $T: \mathbb{R}^n \to \mathbb{R}^n$ affine linear transformation

$$T(x) = \underset{\substack{\uparrow \\ \text{matrix}}}{A}x + \underset{\substack{\uparrow \\ \text{vector}}}{b}$$

Then $m(T(E)) = |T|\, m(E)$
$|T| = |\det(A)|$

Pf: $\Rightarrow$ $\forall n > 0$ integer

$$\mathcal{U}(f) \subset \hat{\mathcal{U}}(f) \subset \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1+\frac{1}{n} \end{pmatrix} \mathcal{U}(f)}_{R_n(f)}$$

$\bigcap_{n=1}^{\infty} R_n(f)$ is meas

$$m(\cap R_n(f)) = \lim m(R_n(f))$$
$$= \lim (1+\tfrac{1}{n})\, m(\mathcal{U}(f))$$
$$= m(\mathcal{U}(f))$$

$\rightsquigarrow$ f is meas.

Prop: If $f_n: \mathbb{R} \to [0, \infty)$ seq of integrable f_n, $f_n \searrow f$ a.e. then $\int f_n \searrow \int f$

Pf: $\int f = m(\mathcal{U}(f)) = m(\hat{\mathcal{U}}(f)) \swarrow m(\hat{\mathcal{U}}(f_n))$
$= m(\mathcal{U}(f_n)) = \int f_n$

If a_n bounded seq

$\overline{a_n} = \sup \{a_m : m \geq n\}$

$\underline{a_n} = \inf \{a_m : m \geq n\}$

If f_n is a seq of functions

$\overline{f_n}(x) = \sup \{f_m(x), m \geq n\}$

$\underline{f_n}(x) = \inf \{f_m(x), m \geq n\}$

Prop: $\mathcal{U}(\overline{f_n}) = \bigcup_{k \geq n} \mathcal{U}(f_k)$

$\hat{\mathcal{U}}(\underline{f_n}) = \bigcap_{k \geq n} \hat{\mathcal{U}}(f_k)$

Thm: Suppose we have a seq of f_n, meas, $f_n \to f$ a.e.

$\exists g : \mathbb{R} \to [0, \infty)$ s.t. $g(x) \geq f_n(x)$ a.e. + $\int g < \infty$

Then $\int f_n \to \int f$

Pf: $\mathcal{U}(f_n) \subseteq \mathcal{U}(g)$ + $m(\mathcal{U}(f_n)) \leq m(\mathcal{U}(g)) < \infty$

Also, $\mathcal{U}(\underline{f_n}) \subseteq \mathcal{U}(f_n) \subset \hat{\mathcal{U}}(\overline{f_n})$

$\underline{f_n} \nearrow f$, $\overline{f_n} \searrow f$ $\Rightarrow \lim \int \underline{f_n} = \int f$,

$\lim \int \overline{f_n} = \int f$