Lecture 8 - 02/10/22

Dominated Convergence Thm:

Suppose

- $f_n \to f$ a.e.
- $g \geq f_n$ $\forall n$
- $\int g < \infty$

Then $\int f_n \to \int f$

Pf:

$\bar{f}_n(x) = \sup_{m \geq n} (f_m(x))$

$\underline{f}_n(x) = \inf_{m \geq n} f_m(x)$

} measurable ($\mathcal{U}\bar{f}_n = \bigcup_{m \geq n} \mathcal{U}f_m$ + $\hat{\mathcal{U}}\underline{f}_n = \bigcap_{m \geq n} \hat{\mathcal{U}}f_m$)

$\underline{f}_n(x) \leq f_n(x) \leq \bar{f}_n(x)$

$g \geq f_n \ \forall n \Rightarrow g \geq \bar{f}_n \Rightarrow \int \bar{f}_n \leq \int g < \infty$

$\forall x$ $\lim \underline{f}_n(x) = \lim \inf f_n(x) = \lim \sup f_n(x) = \lim \bar{f}_n(x)$. Thus $\underline{f}_n \nearrow f$ and $\bar{f}_n \searrow f$

So $\int \underline{f}_n \to \int f$ and $\int \bar{f}_n \to \int f$ by monotone convergence thms

Claim: If $f_n \to f$ a.e., + $\int f_n < M$, $\int f < M$ (but $\int f_n \not\to \int f$)

Example:

f_n :

$f = 0$

$\underline{f}_n(x) = 0$

$\bar{f}_n(x) =$

Fatou's Lemma:

- $f_n : \mathbb{R} \to [0, \infty)$ measurable

Then,
$$\int \liminf f_n \le \liminf \int f_n$$

Pf: $\liminf f_n = \lim \underline{f_n} = f$. By monotone convergence $LHS = \int \lim \underline{f_n} = \lim \int \underline{f_n}$. We have $\underline{f_n} \le f_m \quad \forall m \ge n \quad \int \underline{f_n} \le \int f_m$,

$\leadsto \int \underline{f_n} \le \inf_{m \ge n} \int f_m$

$$\lim_{n \to \infty} \int \underline{f_n} \le \lim_{n \to \infty} \inf_{m \ge n} \int f_m = \liminf \int f_n$$

Thm: If $f, g : \mathbb{R} \to [0, \infty)$ are measurable then $\int f + g = \int f + \int g$

Defn: mesomorphism is a measurability preserving map, 1:1
$f : \mathbb{R}^n \to \mathbb{R}^n$
(sends meas set to meas set, same for inverse)

ex. linear transformation

mesometry is a measure-preserving map

- if $f : \mathbb{R} \to \mathbb{R}$, define
$T_f : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \longmapsto (x, y + f(x))$
- Note T_f has inverse T_{-f}

Thm: T_f is an isometry, $f: \mathbb{R} \to [0, \infty)$ measurable. Let $R = [a, b] \times [0, h] \subseteq \mathbb{R}^2$.

$T_f(R) = \mathcal{U}(f + h \cdot \underbrace{\chi_{[a,b]}}) - \mathcal{U}(f)$

1 if $x \in [a,b]$

0 otherwise

Pf: Let $g = h \cdot \chi_{[a,b]}$. $\hat{\mathcal{U}}(f+g) = T_f(\hat{\mathcal{U}} g)$

$= T_g(\hat{\mathcal{U}} f)$. Claim: T_g preserves measurability and measure.

If $f = f \cdot \chi_{[a,b]}$,

$\mathcal{U} f \sqcup T_f(R) = T_g(\mathcal{U} f) \sqcup R$

$\rightsquigarrow$

$m(\mathcal{U} f) + m(T_f R) = m(T_g \mathcal{U} f) + m(R)$

$= m(\mathcal{U} f) + m(R)$

$m(T_f R) = m(R)$

Claim: T_f never increases outer measure.

$\forall A \subset \mathbb{R}^2 \quad \forall \varepsilon > 0 \quad \exists \{R_i\}$ s.t.

$\sum_i m^*(R_i) \leq m^*(A) + \varepsilon$

$\rightsquigarrow \quad m^*(T_f A) \leq \sum_i m^*(T_f R_i)$

$= \sum_i m^*(R_i)$

$\leq m^*(A) + \varepsilon$

Hence $m^*(T_f A) \leq m^*(A)$

Let $T_{-f} = \varphi \circ T_f \circ \varphi$, $\varphi: (x, y) \mapsto (x, -y)$

ψ is measometry, so T_{-f} preserves measurability. Thus $m^*(T_{-f}(A)) \le m^*(A)$

So

$$m^*(A) = m^*(T_f T_{-f} A) \le m^*(T_{-f} A) \le m^*(A)$$

$\therefore T_f$ preserves outer measure so T_f preserves measurability + measure

Pf of $\int f+g = \int f + \int g$

$$m(\mathcal{U}(f+g)) = m(T_f \mathcal{U}_g \sqcup \mathcal{U}_f) = m(T_f \mathcal{U}_g) + m(\mathcal{U}f) = m(\mathcal{U}_g) + m(\mathcal{U}f)$$

Corr: If $\{f_n\}$ is a seq of integrable f_n then

$$\int \underbrace{\sum_{k=1}^{\infty} f_k}_{F} = \sum_{k=1}^{\infty} \int f_k$$

Pf: Let $F_k = \sum_{i=1}^{k} f_i$, $F_k \nearrow F$

$$\int F_k = \sum_{i=1}^{k} \int f_i \quad \text{so} \quad \lim_{k \to \infty} \int F_k = \sum_{k=1}^{\infty} \int f_i$$