Lecture 9: 02/15/22

Defn: Let $\Omega \subseteq \mathbb{R}^m$ be measurable, $f: \Omega \to \mathbb{R}^n$ function. If $\forall$ $V \subseteq \mathbb{R}^n$ open, $f^{-1}(V)$ is measurable f is a measurable function.

Prop: $f: \Omega \to \mathbb{R}^n$ continuous $\Rightarrow$ f measurable

Pf: by continuity $f^{-1}(V)$ is open in Ω. Then $f^{-1}(V)$ is an intersection of an open $V \subseteq \mathbb{R}^m$ and Ω, an intersection of 2 measurable sets

- can just check open boxes, since open set = countable union of open boxes

Prop: f measurable, g continuous $\Rightarrow$ $f \circ g$ measurable

Pf: Let V be open
$(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$

$\Rightarrow$ measurable since $g^{-1}(V)$ is open

Lemma: $f: \Omega \to \mathbb{R}$ measurable iff $\forall$ $a \in \mathbb{R}$ $f^{-1}((a, \infty))$ is measurable

Pf: Open sets in $\mathbb{R}$ are countable union of open intervals. Suffices to show that all open intervals (a,b) have pre-image measurable. Can show $f^{-1}((a,b]))$ is measurable, and approximate open intervals by $(a,b) = \bigcup_n (a, b - 1/n]$.

Defn: A function $f: \Omega \to \mathbb{R}$ measurable is simple if it takes value on a finite subset of $\mathbb{R}$.

- simple functions form a vector space
- can be written as linear combo of characteristic functions
- non-negative measurable function f admits a sequence of simple functions f_n, non-negative and $f_n \leq f_{n+1}$ s.t. $f_n \to f$ point wise
- integration for simple functions is a linear map from the vector space of simple functions $\to \mathbb{R}$

Defn: for f non-negative, measurable

$$\int f = \sup \{ \int s : 0 \leq s \leq f, \ s \text{ simple} \}$$