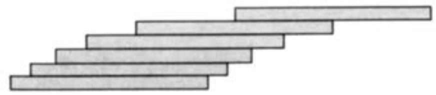
16. MISCELLANEOUS PROBLEMS

1. (a) Show that it is possible to stack a pile of identical books so that the top book is as far as you like to the right of the bottom book. Start at the top and each time place the pile already completed on top of another book so that the pile is just at the point of tipping. (In practice, of course, you can't let them overhang quite this much without having the stack topple. Try it with

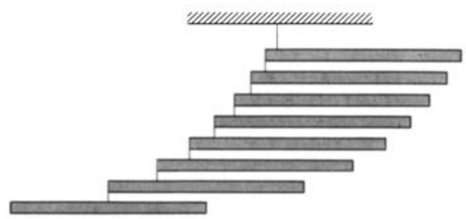


a deck of cards.) Find the distance from the right-hand end of each book to the right-hand end of the one beneath it. To find a general formula for this distance, consider the three forces acting on book n, and write the equation for the torque about its right-hand end. Show that the sum of these setbacks is a divergent series (proportional to the harmonic series). [See "Leaning Tower of *The Physical Reviews,*" Am. J. Phys. **27**, 121–122 (1959).]

(b) By computer, find the sum of N terms of the harmonic series with $N = 25$, 100, 200, 1000, 10^6, 10^{100}.

(c) From the diagram in (a), you can see that with 5 books (count down from the top) the top book is completely to the right of the bottom book, that is, the overhang is slightly over one book. Use your series in (a) to verify this. Then using parts (a) and (b) and a computer as needed, find the number of books needed for an overhang of 2 books, 3 books, 10 books, 100 books.

2. The picture is a mobile constructed of dowels (or soda straws) connected by thin threads. Each thread goes from the left-hand end of a rod to a point on the rod below. Number the rods from the bottom and find, for rod n, the distance from its left end to the thread so that all rods of the mobile will be horizontal. *Hint:* Can you see the relation between this problem and Problem 1?



3. Show that $\sum_{n=2}^{\infty} 1/n^{3/2}$ is convergent. What is wrong with the following "proof" that it diverges?

$$\frac{1}{\sqrt{8}} + \frac{1}{\sqrt{27}} + \frac{1}{\sqrt{64}} + \frac{1}{\sqrt{125}} + \cdots > \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{36}} + \frac{1}{\sqrt{81}} + \frac{1}{\sqrt{144}} + \cdots$$

which is

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \cdots = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right).$$

Since the harmonic series diverges, the original series diverges. *Hint:* Compare $3n$ and $n\sqrt{n}$.

Test for convergence:

4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

5. $\displaystyle\sum_{n=2}^{\infty} \frac{(n-1)^2}{1+n^2}$

6. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n-1}}{(n+1)^2-1}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n^3)}$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{2n^3}{n^4-2}$

Find the interval of convergence, including end-point tests:

9. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\ln(n+1)}$ **10.** $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2 x^n}{(2n)!}$ **11.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2n-1}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n n^2}{5^n(n^2+1)}$ **13.** $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{(-3)^n\sqrt{n}}$

Find the Maclaurin series for the following functions.

14. $\cos[\ln(1+x)]$ **15.** $\ln\left(\dfrac{\sin x}{x}\right)$ **16.** $\dfrac{1}{\sqrt{1+\sin x}}$

17. $e^{1-\sqrt{1-x^2}}$ **18.** $\arctan x = \displaystyle\int_0^x \frac{du}{1+u^2}$

Find the first few terms of the Taylor series for the following functions about the given points.

19. $\sin x,\ a=\pi$ **20.** $\sqrt[3]{x},\ a=8$ **21.** $e^x,\ a=1$

Use series you know to show that:

22. $1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\cdots=\dfrac{\pi}{4}.$ *Hint:* See Problem 18.

23. $\dfrac{\pi^2}{3!}-\dfrac{\pi^4}{5!}+\dfrac{\pi^6}{7!}-\cdots=1$ **24.** $\ln 3+\dfrac{(\ln 3)^2}{2!}+\dfrac{(\ln 3)^3}{3!}+\cdots=2$

25. Evaluate the limit $\lim_{x\to 0} x^2/\ln\cos x$ by series (in your head), by L'Hôpital's rule, and by computer.

Use Maclaurin series to do Problems 26 to 29 and check your results by computer.

26. $\displaystyle\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{1-\cos^2 x}\right)$ **27.** $\displaystyle\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$

28. $\displaystyle\lim_{x\to 0}\left(\frac{1+x}{x}-\frac{1}{\sin x}\right)$ **29.** $\displaystyle\left.\frac{d^6}{dx^6}(x^4 e^{x^2})\right|_{x=0}$

30. (a) It is clear that you (or your computer) can't find the sum of an infinite series just by adding up the terms one by one. For example, to get $\zeta(1.1) = \sum_{n=1}^{\infty} 1/n^{1.1}$ (see Problem 15.22) with error < 0.005 takes about 10^{33} terms. To see a simple alternative (for a series of positive decreasing terms) look at Figures 6.1 and 6.2. Show that when you have summed N terms, the sum R_N of the rest of the series is between $I_N = \int_N^{\infty} a_n\,dn$ and $I_{N+1} = \int_{N+1}^{\infty} a_n\,dn$.

(b) Find the integrals in (a) for the $\zeta(1.1)$ series and verify the claimed number of terms needed for error < 0.005. *Hint*: Find N such that $I_N = 0.005$. Also find upper and lower bounds for $\zeta(1.1)$ by computing $\sum_{n=1}^{N} 1/n^{1.1} + \int_N^{\infty} n^{-1.1}\,dn$ and $\sum_{n=1}^{N} 1/n^{1.1} + \int_{N+1}^{\infty} n^{-1.1}\,dn$ where N is far less than 10^{33}. *Hint*: You want the difference between the upper and lower limits to be about 0.005; find N so that term $a_N = 0.005$.

31. As in Problem 30, for each of the following series, find the number of terms required to find the sum with error < 0.005, and find upper and lower bounds for the sum using a much smaller number of terms.

(a) $\displaystyle\sum_{1}^{\infty} \frac{1}{n^{1.01}}$ (b) $\displaystyle\sum_{1}^{\infty} \frac{1}{n(1+\ln n)^2}$ (c) $\displaystyle\sum_{3}^{\infty} \frac{1}{n\ln n(\ln\ln n)^2}$