Along the circle C', $z = a + \rho e^{i\theta}$, $dz = \rho i e^{i\theta}\,d\theta$, and (3.6) becomes

$$\begin{aligned}\oint_C \phi(z)\,dz = \oint_{C'} \phi(z)\,dz &= \oint_{C'} \frac{f(z)}{z-a}\,dz \\ &= \int_0^{2\pi} \frac{f(z)}{\rho e^{i\theta}} \rho i e^{i\theta}\,d\theta = \int_0^{2\pi} f(z) i\,d\theta.\end{aligned} \tag{3.7}$$

Since our calculation is valid for any (sufficiently small) value of ρ, we shall let $\rho \to 0$ (that is, $z \to a$) to simplify the formula. Because $f(z)$ is continuous at $z = a$ (it is analytic inside C), $\lim_{z\to a} f(z) = f(a)$. Then (3.7) becomes

$$\oint_C \phi(z)\,dz = \oint_C \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} f(z) i\,d\theta = \int_0^{2\pi} f(a) i\,d\theta = 2\pi i f(a) \tag{3.8}$$

or

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\,dz, \qquad a \text{ inside } C. \tag{3.9}$$

This is Cauchy's integral formula. Note carefully that the point a is inside C; if a were outside C, then $\phi(z)$ would be analytic everywhere inside C and the integral would be zero by Cauchy's theorem. A useful way to look at (3.9) is this: If the values of $f(z)$ are given on the boundary of a region (curve C), then (3.9) gives the value of $f(z)$ at any point a inside C. With this interpretation you will find Cauchy's integral formula written with a replaced by z, and z replaced by some different dummy integration variable, say w:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z}\,dw, \qquad z \text{ inside } C. \tag{3.10}$$

For some important uses of this theorem, see Problems 11.3 and 11.36 to 11.38.

▶ PROBLEMS, SECTION 3

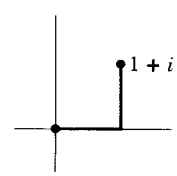
Evaluate the following line integrals in the complex plane by direct integration, that is, as in Chapter 6, Section 8, *not* using theorems from this chapter. (If you see that a theorem applies, use it to check your result.)

1. $\int_i^{i+1} z\,dz$ along a straight line parallel to the x axis.

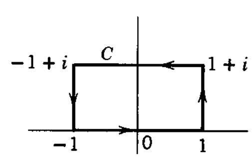
2. $\int_0^{1+i} (z^2 - z)\,dz$

(a) along the line $y = x$;

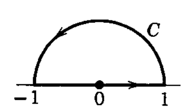
(b) along the indicated broken line.



3. $\oint_C z^2\,dz$ along the indicated paths:



(a)

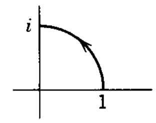


(b)

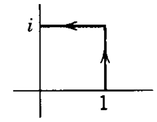
4. $\int dz/(1-z^2)$ along the whole positive imaginary axis, that is, the y axis; this is frequently written as $\int_0^{i\infty} dz/(1-z^2)$.

5. $\int e^{-z}$ along the positive part of the line $y=\pi$; this is frequently written as $\int_{i\pi}^{\infty+i\pi} e^{-z}\,dz$.

6. $\int_1^i z\,dz$ along the indicated paths:



(a)

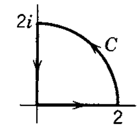


(b)

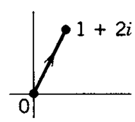
7. $\displaystyle\int \frac{dz}{8i+z^2}$ along the line $y=x$ from 0 to ∞.

8. $\displaystyle\int_{2\pi}^{2\pi+i\infty} e^{2iz}\,dz$ **9.** $\displaystyle\int_{1+2i}^{\infty+2i} \frac{dz}{(x-2i)^2}$ **10.** $\displaystyle\int_2^{2+i\infty} ze^{iz}\,dz$

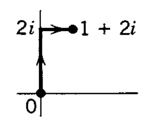
11. Evaluate $\oint_C (\bar{z}-3)\,dz$ where C is the indicated closed curve along the first quadrant part of the circle $|z|=2$, and the indicated parts of the x and y axes. *Hint:* Don't try to use Cauchy's theorem! (Why not? *Further hint:* See Problem 2.3.)



12. $\int_0^{1+2i} |z|^2\,dz$ along the indicated paths:



(a)



(b)

13. In Chapter 6, Section 11, we showed that a necessary condition for $\int_a^b \mathbf{F}\cdot d\mathbf{r}$ to be independent of the path of integration, that is, for $\oint_C \mathbf{F}\cdot d\mathbf{r}$ around a simple closed curve C to be zero, was $\operatorname{curl}\mathbf{F}=0$, or in two dimensions, $\partial F_y/\partial x = \partial F_x/\partial y$. By considering (3.2), show that the corresponding condition for $\oint_C f(z)\,dz$ to be zero is that the Cauchy-Riemann conditions hold.

14. In finding complex Fourier series in Chapter 7, we showed that

$$\int_0^{2\pi} e^{inx}e^{-imx}\,dx = 0, \qquad n \neq m.$$

Show this by applying Cauchy's theorem to

$$\oint_C z^{n-m-1}\,dz, \qquad n > m,$$

where C is the circle $|z|=1$. (Note that although we take $n>m$ to make z^{n-m-1} analytic at $z=0$, an identical proof using z^{m-n-1} with $n<m$ completes the proof for all $n \neq m$.)

15. If $f(z)$ is analytic on and inside the circle $|z|=1$, show that $\int_0^{2\pi} e^{i\theta} f(e^{i\theta})\,d\theta = 0$.

16. If $f(z)$ is analytic in the disk $|z| \le 2$, evaluate $\int_0^{2\pi} e^{2i\theta} f(e^{i\theta})\,d\theta$.

Use Cauchy's theorem or integral formula to evaluate the integrals in Problems 17 to 20.

17. $\displaystyle\oint_C \frac{\sin z\,dz}{2z-\pi}$ where C is the circle (a) $|z|=1$, (b) $|z|=2$.

18. $\displaystyle\oint_C \frac{\sin 2z\,dz}{6z-\pi}$ where C is the circle $|z|=3$.

19. $\oint \frac{e^{3z}\,dz}{z - \ln 2}$ if C is the square with vertices $\pm 1 \pm i$.

20. $\oint_C \frac{\cosh z\,dz}{2\ln 2 - z}$ if C is the circle (a) $|z| = 1$, (b) $|z| = 2$.

21. Differentiate Cauchy's formula (3.9) or (3.10) to get

$$f'(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-z)^2} \qquad \text{or} \quad f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^2}.$$

By differentiating n times, obtain

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-z)^{n+1}} \qquad \text{or} \quad f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^{n+1}}.$$

Use Problem 21 to evaluate the following integrals.

22. $\oint_C \frac{\sin 2z\,dz}{(6z-\pi)^3}$ where C is the circle $|z| = 3$.

23. $\oint_C \frac{e^{3z}\,dz}{(z - \ln 2)^4}$ where C is the square in Problem 19.

24. $\oint_C \frac{\cosh z\,dz}{(2\ln 2 - z)^5}$ where C is the circle $|z| = 2$.

4. LAURENT SERIES

Theorem VII Laurent's theorem [equation (4.1)] (which we shall state without proof). Let C_1 and C_2 be two circles with center at z_0. Let $f(z)$ be analytic in the region R between the circles. Then $f(z)$ can be expanded in a series of the form

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots \tag{4.1}$$

convergent in R. Such a series is called a *Laurent series*. The "b" series in (4.1) is called the *principal part* of the Laurent series.

Example 1. Consider the Laurent series

$$f(z) = 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots + \left(\frac{z}{2}\right)^n + \cdots + \frac{2}{z} + 4\left(\frac{1}{z^2} - \frac{1}{z^3} + \cdots + \frac{(-1)^n}{z^n} + \cdots\right). \tag{4.2}$$

Let us see where this series converges. First consider the series of positive powers; by the ratio test (see Chapters 1 and 2), this series converges for $|z/2| < 1$, that is, for $|z| < 2$. Similarly, the series of negative powers converges for $|1/z| < 1$, that is, $|z| > 1$. Then both series converge (and so the Laurent series converges) for $|z|$ between 1 and 2, that is, in a ring between two circles of radii 1 and 2.

We expect this result in general. The "a" series is a power series, and a power series converges *inside* some circle (say C_2 in Figure 4.1). The "b" series is a series