1. (a) $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ <u>convergent</u>.

$$\frac{1}{\sqrt{x}}\,dx = 2\,d\sqrt{x}.$$

(b) $\int_{\mathbb{R}^2} (1+x^2+y^2)^{-1}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} (1+r^2)^{-1}\, r\,dr\,d\theta$

$= \frac{2\pi}{2}\int_0^{\infty} \frac{1}{1+r^2}\,dr^2$ <u>divergent</u>

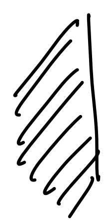
(c) $\left|\int_0^{\infty} e^{(-1+i)x}\,dx\right| \leq \int_0^{\infty} \left|e^{(-1+i)x}\right|\,dx$

$= \int_0^{\infty} e^{-x}\,dx < \infty$

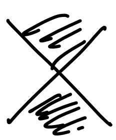
convergent

(d) $\sum_{n=3} (-1)^n \frac{1}{\log n}$ this is convergent, by alternating serives test:

(e) $\sum_n \sin(n)$ divergent. $\lim_n \sin(n) \neq 0$.

2. (a) (1)  $Re(z) \leq 0$ $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$

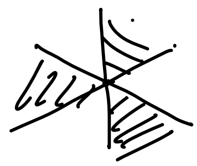
(2) $Re(z^2) \leq 0$, $2\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) + 2n\pi$

 $\Leftrightarrow$ $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4}) + n\pi$

(3) $Re(z^3) \leq 0$

$$\Leftrightarrow \quad 3\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) + 2\pi n$$

$$\theta \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right) + \frac{2\pi n}{3}$$



(b). let $u = Re(f) = \frac{x}{x^2+y^2}$, $\quad v = Im(f) = \frac{-y}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x \cdot x}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{-1}{x^2+y^2} + \frac{2y \cdot y}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} \qquad = \frac{-(x^2+y^2) + 2y^2}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2} \qquad = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

$$\frac{\partial u}{\partial y} = -\frac{2y \cdot x}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

$$\therefore \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This satisfies Cauchy-Riemann condition.

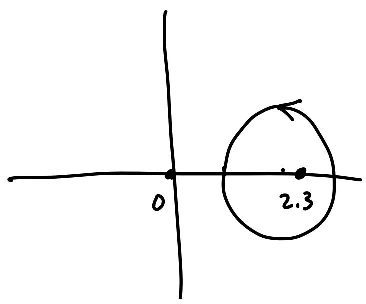
(3). $\frac{1}{z} \frac{1+z}{1-z} = \frac{1}{z}(1+z)(1+z+z^2+\cdots)$

$$= \frac{1}{z}(1+2z+\cdots) = \frac{1}{z} + 2 + \cdots$$

#3 (a) $$\frac{1}{2\pi i}\oint_{|z|=1}\left(az+bz+\frac{c}{z}\right)dz = c$$

(b) $$\frac{1}{2\pi i}\oint_{|z-2|=1}\left(ae^{\frac{1}{z}}+\frac{b}{z-2.3}+\frac{c}{z}\right)dz = b$$

since 2.3 is the only pole inside the contour.



(c) $$\frac{1}{2\pi i}\oint_{|z|=2}\frac{2023\cdot z^4+12\cdot z^3+11\cdot z^2+8z+11}{z^4-1}\,dz$$

the poles are at the roots of $z^4=1$, on the unit circle. $|z|=1$.

$$\oint_{|z|=2}\frac{z^4}{z^4-1}dz = \oint_{|z|=2}\frac{z^4-1+1}{z^4-1}dz = \oint_{|z|=2} 1+\frac{1}{z^4-1}\,dz$$

$$= \oint \frac{1}{z^4-1}dz$$

$k=0,1,2.$ $$\oint_{|z|=2}\frac{z^k}{z^4-1}dz = \lim_{R\to\infty}\oint_{|z|=R}\frac{z^k}{z^4-1}dz = 0$$

$k=3,$ $$\oint_{|z|=2}\frac{z^3}{z^4-1}dz = \lim_{R\to\infty}\oint_{|z|=R}\frac{z^3}{z^4(1-\frac{1}{z^4})}dz$$

$$= \lim_{R\to\infty}\oint_{|z|=R}\frac{1}{z}\left(1+\frac{1}{z^4}+\left(\frac{1}{z^4}\right)^2+\cdots\right)dz$$

$$= 2\pi i$$

$$\therefore \quad \frac{1}{2\pi i} \oint \frac{az^4+bz^3+cz^2+dz+e}{z^4-1} \, dz = b$$

(d) $$\int_0^{2\pi} \frac{(1+e^{i\theta}+e^{2i\theta})^2}{(1-0.1e^{i\theta})} \, d\theta$$

$$= \int_0^{2\pi} (1+e^{i\theta}+e^{2i\theta})^2 \left(1+0.1e^{i\theta}+(0.1e^{i\theta})^2+\cdots\right) d\theta$$

$$= \int_0^{2\pi} \left(1+\sum_{n=1}^{\infty} c_n \cdot e^{in\theta}\right) d\theta = 2\pi.$$

$$\because \int_0^{2\pi} e^{in\theta} \, d\theta = \begin{cases} 2\pi & n=0 \\ 0 & \text{else} \end{cases}$$

#4 (a) $$\int_0^{2\pi} \frac{1}{2+\cos x} \, dx = \int_0^{2\pi} \frac{1}{2+\left(\frac{e^{ix}+e^{-ix}}{2}\right)} \, dx$$

let $z=e^{ix}$, then $dx = \frac{dz}{iz}$, z runs on $\{|z|=1\}$

$$\oint_{|z|=1} \frac{1}{2+\left(\frac{z+z^{-1}}{2}\right)} \frac{dz}{iz} = \frac{1}{2i} \oint_{|z|=1} \frac{1}{z^2+4z+1} \, dz$$

$$= \frac{1}{2i} \oint \frac{1}{(z+2)^2-3} \, dz$$

$1<\sqrt{3}<2$

there are 2 roots. $z = \sqrt{3}+2$, $\sqrt{3}-2$.

$-1<\sqrt{3}-2<0$, $\sqrt{3}+2>1$.

Hence there is one pole at $z = \sqrt{3}+2$ inside the contour

$$\frac{1}{2i} \oint \frac{1}{(z+2)^2-3}\, dz = \frac{1}{2i} \cdot 2\pi i \cdot \operatorname{Res}_{z=\sqrt{3}-2} \frac{1}{(z+2)^2-3}$$

$$= \pi \frac{1}{2(\sqrt{3})} = \boxed{\frac{\pi}{2\sqrt{3}}}$$

(b) $$I = \int_{\mathbb{R}} \overbrace{\frac{e^{ix}}{(x+i)(x-2i)(x+3i)}}^{f(x)}\, dx$$

to ensure convergence, we ~~use~~ close the contour using upper semi-circle

$$I = \lim_{R\to\infty} \left(\int_{-R}^{R} f(z)\, dz + \oint_{\substack{z=e^{i\theta}\cdot R \\ \theta\in(0,\pi)}} f(z)\, dz \right)$$

$$= \oint_{-R}^{R} \frac{e^{iz}}{(z+i)(z-2i)(z+3i)}\, dz = 2\pi i \cdot \frac{e^{-2}}{(2i+i)(2i+3i)}$$

Fix $R \gg 1$. pick up pole at $z=2i$ $$= \frac{2\pi i \cdot e^{-2}}{-15}$$

#5 (a) $$f(x) = \frac{1}{1-\frac{1}{2}e^{ix}} = 1 + \frac{1}{2}e^{ix} + \left(\frac{1}{2}e^{ix}\right)^2 + \cdots\cdots$$

$$F_n = \begin{cases} 0 & n<0 \\ \left(\frac{1}{2}\right)^n & n \geq 0 \end{cases}$$

(b) $g(x) = \dfrac{1}{1-2e^{ix}} = \dfrac{-1}{2e^{ix}} \quad \dfrac{1}{1-\frac{1}{2}e^{-ix}}$

$$= \frac{-1}{2e^{ix}}\left(1+\tfrac{1}{2}e^{-ix} + \left(\tfrac{1}{2}e^{-ix}\right)^2 + \cdots\right)$$

$$= \left(-\tfrac{1}{2}\right)e^{-ix} + \left(-\tfrac{1}{2^2}\right)e^{-ix\cdot 2} + \cdots$$

$$G_n = \begin{cases} 0 & n \geq 0 \\ -\frac{1}{2^{|n|}} = -2^n & n < 0 \end{cases}$$

#6 : • If we use answer from #5, then

$$\langle f, g\rangle = \Sigma_n F_n \cdot \overline{G_n} = 0$$

$$(f*g)(x) = \Sigma_n F_n \cdot G_n \cdot e^{inx} = 0$$

• if we start from scratch, we get

(a) $\langle f, g\rangle = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \dfrac{1}{1-\frac{1}{2}e^{ix}} \cdot \overline{\left(\dfrac{1}{1-2\cdot e^{ix}}\right)}\, dx$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-\frac{1}{2}e^{ix}} \quad \frac{1}{1-2e^{-ix}}\, dx$$

let $z = e^{ix}$

$$= \frac{1}{2\pi}\oint_{|z|=1} \frac{1}{1-\frac{1}{2}z} \quad \frac{1}{1-2/z} \quad \frac{dz}{iz}$$

$$= \frac{1}{2\pi i} \oint_{|z|=1} \frac{-1}{(z-2)^2}\, dz = 0$$

(b) $$(f * g)(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1-\frac{1}{2}e^{iu}} \frac{1}{1-2e^{i(x-u)}}\, du$$

$(z = e^{iu})$ $$= \frac{1}{2\pi} \oint_{|z|=1} \frac{1}{1-\frac{1}{2}z} \frac{1}{1-2e^{ix}/z} \frac{dz}{iz}$$

$$= \frac{1}{\pi i} \oint_{|z|=1} \frac{-1}{z-2} \cdot \frac{1}{z-2e^{ix}}\, dz$$

the 2 poles are at $z=2$ and $z = 2 \cdot e^{ix}$.

there is no pole inside $|z|=1$.

$= 0$

#7. We can first find a particular sol'n.

$y_p(x) = -1$

The general sol'n to the homoge equation is

$y_h'' - y_h = 0$ $\qquad y_h = c_1 e^{x} + c_2 \cdot e^{-x}$.

Thus $\quad y = y_h + y_p = -1 + c_1 e^{x} + c_2 e^{-x}$

using boundary conditi. we get

$$\begin{cases} -1 + C_1 + C_2 = 0 \\ C_1 - C_2 = 1 \end{cases}$$

$\therefore \quad C_1 = 1, \quad C_2 = 0.$

$$y = -1 + e^x$$

- Or, we use Laplace method:

$LT(y) = Y$

$LT(y') = pY - y(0) = pY$

$LT(y'') = p \cdot LT(y') - y'(0) = p(pY - y(0)) - y'(0)$

$= p^2 Y - 1$

$LT(1) = 1/p$

thus

$$(p^2 Y - 1) - Y = \frac{1}{p}$$

$$\therefore \quad Y = \left(1 + \frac{1}{p}\right) / (p^2 - 1)$$

$$= \frac{p+1}{p(p-1)(p+1)} = \frac{1}{p(p-1)}$$

$$= \frac{1}{p-1} - \frac{1}{p}$$

$$LT^{-1}\left(\frac{1}{p}\right) = 1, \qquad LT^{-1}\left(\frac{1}{p-1}\right) = e^x.$$

thus $\quad y(x) = e^x - 1.$

#8. $(D^2-1)f(x) = \delta(x-a)$ $\qquad f(0)=0,\ f(1)=0.$

the general sol'n to the homogeneous eq.

$$(D^2-1)f(x) = 0$$

is $e^{\pm x}$

let $f(x) = \begin{cases} f_-(x) & 0<x<a \\ f_+(x), & 1>x>a \end{cases}$

then $f_\pm(x)$ solves the homog. equation

considering boundary condition, we get

$$f(x) = \begin{cases} A\sinh(x) & 0<x<a \\ B\sinh(x-1) & a<x<1 \end{cases}$$

Then $\begin{cases} f_+(a) = f_-(a) \\ f_+'(a) - f_-'(a) = 1 \end{cases}$

$$\Leftrightarrow \begin{cases} A\cdot\sinh(a) = B\sinh(a-1) \\ -A\cosh(a) + B\cosh(a-1) = 1 \end{cases}$$

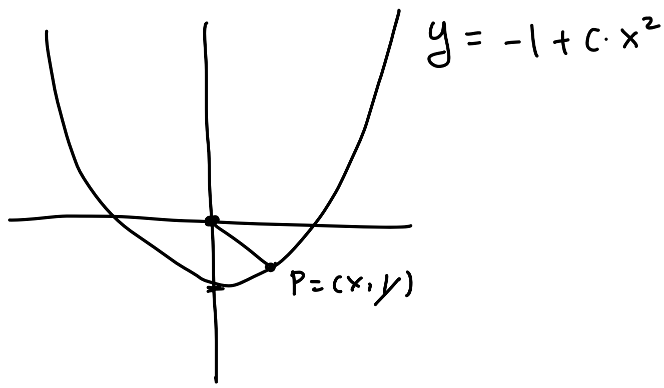
$$\therefore \quad A = \frac{\sinh(a-1)}{-\cosh(a)\sinh(a-1) + \sinh(a)\cosh(a-1)} = \frac{\sinh(a-1)}{\sinh 1}$$

$$B = \frac{\sinh(a)}{-\cosh(a)\sinh(a-1) + \sinh(a)\cosh(a-1)} = \frac{\sinh(a)}{\sinh(1)}$$

(we can use $\cosh(a)\sinh(b) - \sinh(a)\cosh(b) = \sinh(b-a)$

to simplify the denominator to $\sinh(a-(a-1)) = \sinh(1)$.)

#9:



We use Lagrange multiplier to figure out the extremizer.

To minimize $\sqrt{x^2+y^2}$ under the constraint $y=-1+cx^2$ is the same as minimizing x^2+y^2, with the same constraint.

$$I = x^2+y^2 - \lambda(y+1-cx^2)$$

$$\frac{\partial I}{\partial x} = 2x + 2\lambda c\cdot x = (1+c\lambda)2x$$

$$\frac{\partial I}{\partial y} = 2y - \lambda$$

$$\therefore \begin{cases} \frac{\partial I}{\partial x} = 0 \\ \frac{\partial I}{\partial y} = 0 \end{cases} \iff \begin{cases} y = \frac{\lambda}{2} \\ x = 0, \end{cases} \text{ or } \begin{cases} y = \lambda/2 \\ \lambda = -\frac{1}{c}. \end{cases} \quad x \text{ free}$$

plug into the constraint $y = -1 + cx^2$

we get

$$\begin{cases} x = 0 \\ y = -1 \\ \lambda = -2 \end{cases} \quad \text{or} \quad \begin{cases} \lambda = -\frac{1}{c} \\ y = -\frac{1}{2c}. \\ x = \pm\sqrt{\frac{y+1}{c}} \end{cases}$$

For $c = \frac{1}{4}$. the 2nd sol'n is not presents

only sol'n is $(x=0, y=-1)$.

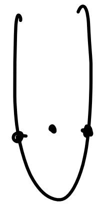
For $c = 4$, we have

$$\begin{cases} x = 0 \\ y = -1 \end{cases} \qquad \begin{cases} y = -\frac{1}{8} \\ x = \pm\sqrt{\frac{7/8}{4}} = \pm\sqrt{\frac{7}{32}} \end{cases}$$

$x^2+y^2 = 1$

$$x^2+y^2 = \frac{7}{32} + \frac{1}{64} = \frac{15}{64}$$

$\boxed{\text{minimizer}}$



#10 :

(a) $I = \int y\sqrt{1+(x')^2}\, dy$

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$$

$$\Leftrightarrow \quad y\frac{x'}{\sqrt{1+(x')^2}} = c$$

$$\Leftrightarrow \quad \frac{x'}{\sqrt{1+(x')^2}} = \frac{c}{y} \quad \Leftrightarrow \quad \frac{(x')^2}{1+(x')^2} = \frac{c^2}{y^2}$$

$$\Leftrightarrow \quad (x')^2 = \frac{c^2}{y^2-c^2}$$

$$\Leftrightarrow \quad x' = \frac{1}{\sqrt{\left(\frac{y}{c}\right)^2 - 1}}$$

let $y/c = \cosh u$, then

$$dx = \frac{1}{\sinh u} \, c \cdot \sinh u \, du$$

$$x = c \cdot u + C_2$$

$$\therefore \quad \frac{y}{c} = \cosh\left(\frac{x - x_0}{c}\right)$$

(b) $I = \int (x + y^2 y')^2 \, dx.$

let $u = y^3/3$ then $y^2 y' = u'.$

$$I = \int \underbrace{(x + u')^2}_{=F} \, dx$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) - \frac{\partial F}{\partial u} = 0$$

$$\Leftrightarrow \quad 2(x + u') = \text{const}$$

$$\Leftrightarrow \quad x + u' = C_1$$

$$\frac{du}{dx} = C_1 - x$$

$$u = C_1 x - \frac{1}{2}x^2 + C_2$$

$$\therefore \quad \boxed{\frac{1}{3}y^3 + \frac{1}{2}x^2 = C_1 x + C_2.}$$