Math 121A Final December 11, 2023

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

Please include calculation steps and reasoning (except in problem 1).

1. Determine whether each integral or series is convergent or divergent. No explanation needed.

   (a) $\int_0^1 x^{-1/2}dx$
   (b) $\int_{\mathbb{R}^2}(1+x^2+y^2)^{-1}dxdy$
   (c) $\int_0^\infty e^{(-1+i)x}dx$
   (d) $\sum_{n=2}^\infty(-1)^n(\log n)^{-1}$ (log is natural log)
   (e) $\sum_n \sin(n)$.

2. (a) (3 points) Sketch the following regions in the complex plane $z \in \mathbb{C}$:

$$(1)Re(z) \leq 0, \quad (2)Re(z^2) \leq 0, \quad (3)Re(z^3) \leq 0.$$

   (b) (3 points) Use Cauchy Riemann condition to test whether the following function is analytic on $x > 0, y > 0$

$$f(x,y) = \frac{x - iy}{x^2 + y^2}.$$

   (c) (4 points) Compute the Laurent expansion of

$$\frac{1+z}{z(1-z)}$$

   at $z = 0$, keep the first two non-zero terms.

3. Compute the following integrals, where integral on the circle contour is in counter-clockwise direction.

   (a) (2 pts)

$$\frac{1}{2\pi i}\oint_{|z|=1}(az + b + \frac{c}{z})dz.$$

   (b) (3 pts)

$$\frac{1}{2\pi i}\oint_{|z-2|=1}(ae^{1/z} + \frac{b}{z-2.3} + \frac{c}{z})dz.$$

   (c) (3 pts)

$$\frac{1}{2\pi i}\oint_{|z|=2}\frac{2023z^4 + 12z^3 + 11z^2 + 8z + 11}{z^4 - 1}dz$$

   (d) (2 pts)

$$\int_0^{2\pi}\frac{(1 + e^{i\theta} + e^{2i\theta})^2}{1 - 0.1e^{i\theta}}d\theta$$

4. Compute the following integrals

   (a) (5 pts)

$$\int_0^{2\pi}\frac{1}{2 + \cos(x)}dx$$

(b) (5 pts)

$$\int_{\mathbb{R}} \frac{e^{ix}}{(x+i)(x-2i)(x+3i)} dx$$

5. Let $f : \mathbb{R} \to \mathbb{C}$ be a periodic function with period 2π. The Fourier series expansion of $f(x)$ is

$$f(x) = \sum_{n \in \mathbb{Z}} F_n e^{inx}, \qquad \text{where } F_n = (2\pi)^{-1} \int_0^{2\pi} f(x) e^{-inx} dx$$

(a) (5 points) Let

$$f(x) = \frac{1}{1 - e^{ix}/2}$$

find its Fourier coefficients F_n for all $n \in \mathbb{Z}$.

(b) (5 points) Let

$$g(x) = \frac{1}{1 - 2e^{ix}}$$

find its Fourier coefficients G_n for all $n \in \mathbb{Z}$.

6. Use the same f, g as in problem 5.

(a) Compute the Hermitian inner product $\langle f, g \rangle$ defined as

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} dx$$

(b) Compute their convolution defined as

$$(f \star g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(u) g(x-u) du$$

(Hint: you can (but don't have to) use the answer in problem 5, and use Parseval identity $\langle f, g \rangle = \sum_n F_n \bar{G}_n$, and convolution formula $f \star g = \sum_n F_n G_n e^{inx}$.

7. Solve the following equation over $x \geq 0$.

$$y''(x) - y(x) = 1$$

with boundary condition that

$$y(0) = 0, \quad y'(0) = 1.$$

8. Let $a \in (0, 1)$. Solve the following equation

$$(D^2 - 1) f(x) = \delta(x - a), \quad D = d/dx, \quad x \in [0, 1]$$

with boundary condition that $f(0) = 0, f(1) = 0$.

9. Consider the parabola $H = \{(x, y) \in \mathbb{R}^2 \mid y = cx^2 - 1\}$, where $c > 0$. Consider the two cases, $c = 1/4$ and $c = 4$. In each case, find the point(s) on the parabola, such that the distance to $(0, 0)$ is minimized.

10. Find the Euler-Lagrange equation for the following variational problem and write down the general solution. You may use

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}dx = \sqrt{1 + (dx/dy)^2}dy.$$

   (a) $I = \int yds$.
   (b) $I = \int (x + y^2y')^2dx$