MATH 121A HW2

Lecture 8/28

Great

1. $V \subset \mathbb{R}^3$ defined as $\{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$

a) Find a basis for V $(\underline{e}_1, \dots, \underline{e}_n)$

x_3 is dependent on x_1 and x_2

$x_3 = -x_1 - x_2$

thus, there are 2 deg of freedom $(n = 2)$

$\Rightarrow$ 2 basis vectors $\underline{e}_1, \underline{e}_2$

$\underline{e}_1$: let $x_1 = 1$, $x_2 = 0$

$\Rightarrow x_3 = -1$

$\Rightarrow \underline{e}_1 = (1, 0, -1)$

$\underline{e}_2$: let $x_1 = 0$, $x_2 = 1$

$\Rightarrow x_3 = -1$

$\Rightarrow \underline{e}_2 = (0, 1, -1)$

linearly independent since $\underline{e}_1 \neq c\,\underline{e}_2$ for $c \in \mathbb{R}$

$\therefore$ $\boxed{\begin{aligned} \underline{e}_1 &= (1, 0, -1) \\ \underline{e}_2 &= (0, 1, -1) \end{aligned}}$

b) $(2, -1, -1) = a\underline{e}_1 + b\underline{e}_2 = a(1, 0, -1) + b(0, 1, -1)$

let $a = 2$, solve for b:

then $(2, -1, -1) = (2, 0, -2) + b(0, 1, -1)$

$(0, -1, 1) = b(0, 1, -1)$

$\Rightarrow b = -1$

$\therefore$ $\boxed{(2, -1, -1) = 2\underline{e}_1 - \underline{e}_2}$ ✓

(2.) Let $W = \mathbb{R}^2$, $V \to W$ the map of forgetting coordinate x_3 :
(Assume V is still as defined previously)
i.e. for $\underline{v} = (x_1, x_2, x_3)$ in $V \subset \mathbb{R}^3$, then
$f: V \to W$ maps to $\underline{w} = (x_1, x_2)$ in $W = \mathbb{R}^2$

1. <u>Check if $f: V \to W$ is a linear map</u>

let $\underline{v}_1 = (x_1, y_1, -x_1 - y_1) \in V$
$\underline{v}_2 = (x_2, y_2, -x_2 - y_2) \in V$

then
$$\begin{aligned} f(\underline{v}_1 + \underline{v}_2) &= f(x_1 + x_2, y_1 + y_2, -x_1 - y_1 - x_2 - y_2) \\ &= (x_1 + x_2, y_1 + y_2) \\ &= (x_1, y_1) + (x_2, y_2) \\ &= f(\underline{v}_1) + f(\underline{v}_2) \checkmark \end{aligned}$$

$\Rightarrow$ f preseves vecter additin

let $c \in \mathbb{R}$
then
$$\begin{aligned} f(c\underline{v}_1) &= f(cx_1, cy_1, -cx_1 - cy_1) \\ &= (cx_1, cy_1) \\ &= c(x_1, y_1) = cf(\underline{v}_1) \checkmark \end{aligned}$$

$\Rightarrow$ f preseves scaler multiplicatin

thus, $f: V \to W$ is a <u>linear map</u>

2. <u>Check if $f: V \to W$ is injective</u>

Injective $\Rightarrow$ different vecters in V map to different vecters in W (one-to-one)

- Since $V = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$, x_3 is always dependent on x_1 and x_2 $(x_3 = -x_1 - x_3)$
- Thus, for all $\underline{v}_1, \underline{v}_2 \in V$, $\underline{v}_1 \neq \underline{v}_2$ iff $x_1 \neq x_2$ <u>or</u> $y_1 \neq y_2$

2. cont.

- Consequently, $f(\underline{v}_1) = f(\underline{v}_2) \iff \underline{v}_1 = \underline{v}_2$ because the x_3 value that is dropped in the transformation is fully constrained by x_1 and x_2 and cannot take on different values for the same x_1 and x_2
- Thus, different vectors in V will always map to different vectors in $W \Rightarrow f: V \to W$ is <u>injective</u>

3. <u>Check if $f: V \to W$ is surjective</u>

Surjective $\Rightarrow$ the range of V covers all possible values in W (onto)

$\underline{v} = (x_1, x_2, x_3) \in V$, where $x_1, x_2 \in \mathbb{R}$ and $x_3 = -x_1 - x_2$

- thus, when x_3 is dropped, the remaining (x_1, x_2) can take on any value in $\mathbb{R}^2$
- so $f: V \to W$
  $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ maps to all $\mathbb{R}^2$

$\Rightarrow f: V \to W$ is <u>surjective</u>

$\therefore$ $f: V \to W$ must be an <u>isomorphism</u> given that it is linear, injective, and surjective

<u>What is the inverse?</u>

Find $f^{-1}: W \to V$

given $\underline{w} = (x_1, x_2) \in W = \mathbb{R}^2$

map to $\underline{v} = (x_1, x_2, x_3) \in V \subset \mathbb{R}^3$,

where $x_1 + x_2 + x_3 = 0 \Rightarrow x_3 = -x_1 - x_2$

(x_1 and x_2 are preserved in either direction)

2. Thus, inverse is

$$\boxed{\begin{aligned} f^{-1}: W &\to V \\ (x_1, x_2) &\longmapsto (x_1, x_2, -x_1 - x_2) \end{aligned}}$$

3. $V = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$ (same as before)

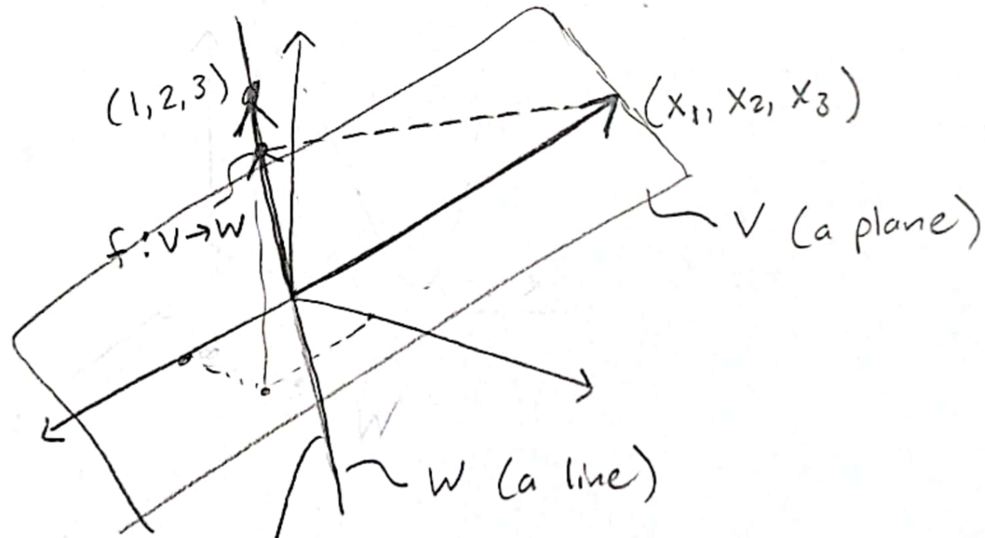
Let W be the line generated by the vector $(1, 2, 3)$

$\Rightarrow W = \{c(1,2,3)\}$ for $c \in \mathbb{R}$

$f: V \to W$ is orthogonal projection

maps $\underline{v} \in V$ to closest point on W

Is f a linear map?



- In this case, $\underline{w} = (1,2,3) \in W$ is a basis for the subspace W (which is a line)
- Then the orthogonal projection of some vector $\underline{v} \in V$ onto the line W is given by:

$$\hat{\underline{v}} = \text{proj}_W \underline{v} = \frac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \underline{w}$$

3. cont.

Hence the transformation is $f(\underline{v}) = \frac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \underline{w}$

where $\underline{v} \cdot \underline{w} = \langle (x_1, x_2, x_3), (1, 2, 3) \rangle$ is the standard inner product

a) Show that this is a linear map:

1. Preserves vector addition?

let $\underline{v}_1, \underline{v}_2 \in V$

$$f(\underline{v}_1 + \underline{v}_2) = \frac{(\underline{v}_1 + \underline{v}_2) \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \underline{w} = \frac{\underline{v}_1 \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \underline{w} + \frac{\underline{v}_2 \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \underline{w}$$

$$= f(\underline{v}_1) + f(\underline{v}_2)$$

(since standard inner product distributes across addition)

$\Rightarrow$ yes, preserves vector addition

2. Preserves scalar multiplication?

let $c \in \mathbb{R}$, $\underline{v} \in V$

$$f(c\underline{v}) = \frac{(c\underline{v}) \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \underline{w} = c\left(\frac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \underline{w}\right) = c f(\underline{v})$$

Again using properties of standard inner product

$\Rightarrow$ yes, preserves scalar multiplication

$\therefore$ The orthogonal projection $f: V \to W$ is a linear map

b) What is the kernel of f?

$\ker(f) = \underline{v}$ in V s.t. $f(\underline{v}) = \frac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \underline{w} = \underline{0}$ in W

where $\text{proj}_{\underline{w}} \underline{v} = \frac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \underline{w} = \underline{0} \Rightarrow (\underline{v} \cdot \underline{w}) \underline{w} = \underline{0}$

$\Rightarrow \underline{v} \cdot \underline{w} = \underline{0}$

$\Rightarrow (x_1, x_2, x_3) \cdot (1, 2, 3) = 0$

$\Rightarrow x_1 + 2x_2 + 3x_3 = 0$

3. cont.

b) Thus, $Ker(f) = \{(x_1, x_2, x_3)\}$ s.t.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 & (1) \\ x_1 + x_2 + x_3 = 0 & (2) \end{cases}$$

1 deg of freedom

$\downarrow$

should only depend on 1 variable

Subtract $(1) - (2) \Rightarrow \quad x_2 + 2x_3 = 0$

$$x_2 = -2x_3$$

and $x_1 = -x_2 - x_3 = 2x_3 - x_3 = x_3$

$\Rightarrow \quad Ker(f) = \{(x_3, -2x_3, x_3)\}$

Or in a more general sense:

$$\boxed{Ker(f) = \{(x, -2x, x) \in V\}}$$

c) Let $g: W \to V$ be the orthogonal projection in the other direction

let $\underline{w} \in W$ (can be any vector in W this time)

and let $\{\underline{u}_1, \underline{u}_2\} \in V$ be an orthogonal basis for vector space V

Then the orthogonal projection of $\underline{w}$ onto plane V is given by:

$$\hat{\underline{w}} = \text{proj}_V \underline{w} = \frac{\underline{w} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1 + \frac{\underline{w} \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2} \underline{u}_2 \equiv g(\underline{w})$$

- Check if $g(\underline{w}) = \text{proj}_V \underline{w}$ preserves vector addition and scalar multiplication!

let $\underline{w}_1, \underline{w}_2 \in W$ and $c \in \mathbb{R}$

then $$g(c\underline{w}_1 + c\underline{w}_2) = \frac{(c\underline{w}_1 + c\underline{w}_2) \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1 + \frac{(c\underline{w}_1 + c\underline{w}_2) \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2} \underline{u}_2$$

3. cont.

c)
$$= \frac{c\underline{w}_1 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\underline{u}_1 + \frac{c\underline{w}_2 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\underline{u}_1 + \frac{c\underline{w}_1 \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2}\underline{u}_2 + \frac{c\underline{w}_2 \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2}\underline{u}_2$$

$$= c\left(\frac{\underline{w}_1 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\underline{u}_1 + \frac{\underline{w}_2 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\underline{u}_1\right) + c\left(\frac{\underline{w}_1 \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2}\underline{u}_2 + \frac{c\underline{w}_2 \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2}\underline{u}_2\right)$$

$$= cg(\underline{w}_1) + cg(\underline{w}_2)$$

$\Rightarrow$ preserves both vec. addition and scalar mult.

$\therefore$ <u>$g: W \to V$ is a linear map</u>

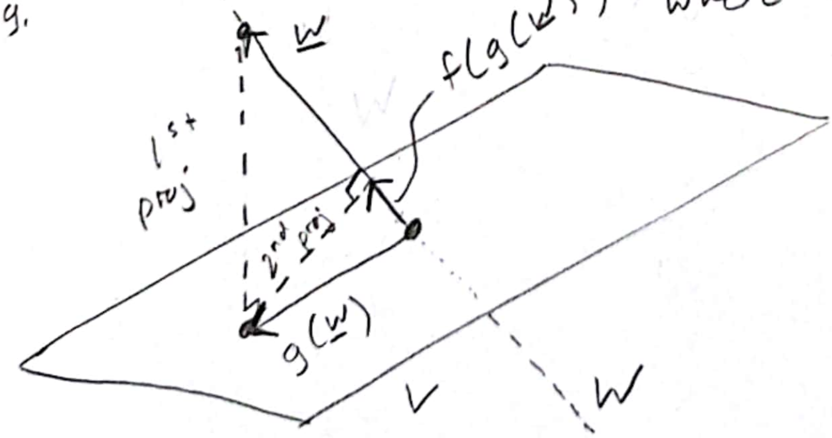
d) f is related to g in the sense that

$f(g(\underline{w}))$ maps $W \to V \to W$ and

$g(f(\underline{v}))$ maps $V \to W \to V$

However, they aren't necessarily perfect inverses (i.e. $f(g(\underline{w})) \neq \underline{w}$ and $g(f(\underline{v})) = \underline{v}$) because the original vector will not be preserved when you do an orthogonal projection back to the original subspace

e.g. where $f(g(\underline{w})) \neq \underline{w}$

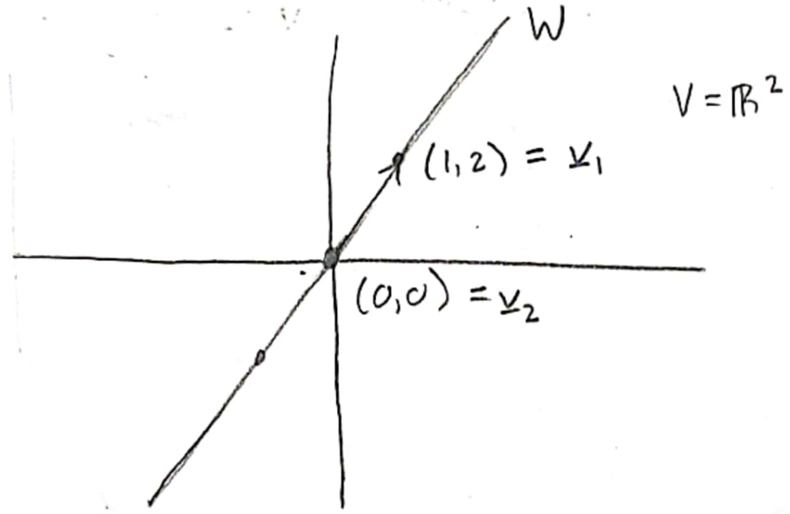


Lecture 8/30

4. $V = \mathbb{R}^2$, W = subspace generated by span$\{(1,2)\}$

For $v \in V$, let $[v] = v + W \in V/W$

a)



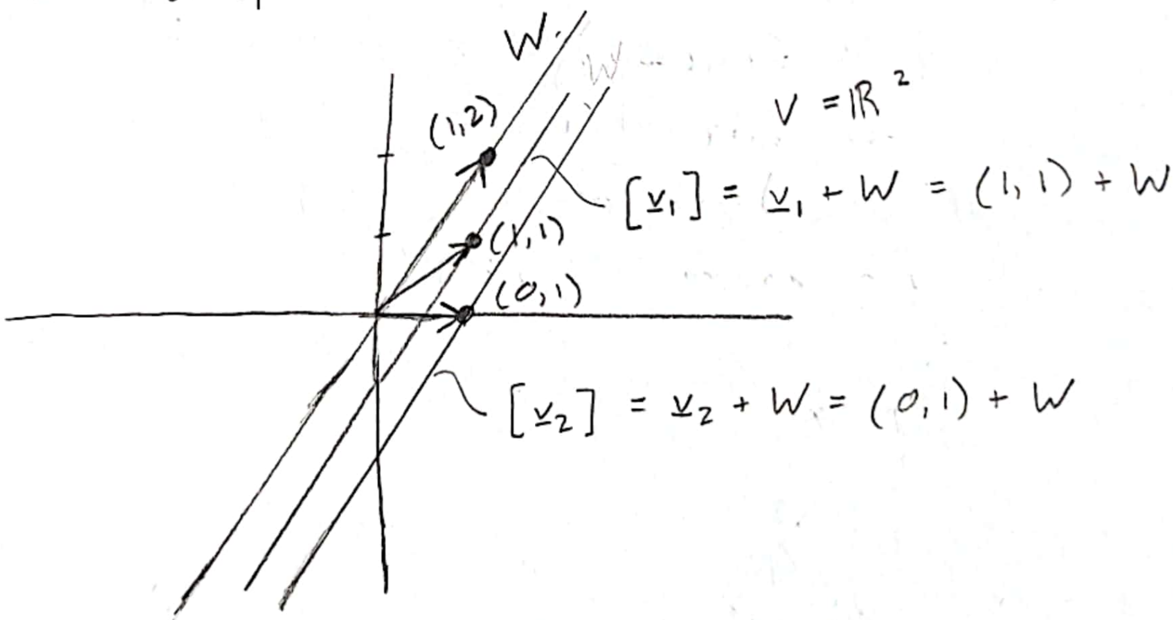
for $\underline{v}_1, \underline{v}_2 \in V$

$\underline{v}_1 - \underline{v}_2 \in W$

$\Rightarrow [\underline{v}_1] = [\underline{v}_2]$

Yes, $[(0,0)] = [(1,2)]$ because $(1,2) - (0,0) = (1,2) \in W$, which means they both lie in subspace W and must be equivalent.

b)



No, $[(1,1)] \neq [(0,1)]$ because $(1,1) - (0,1) = (1,0) \notin W$. As shown, $[(1,1)]$ and $[(0,1)]$ result in 2 separate affine lines, so have different equivalence classes

(5.) Dual vector space $V^* = \text{Hom}(V, \mathbb{R})$

Let $V = \{ax^3 + bx^2 + cx + d \mid a, b, c, d \in \mathbb{R}\}$

(polynomials with degree ≤ 3)

a) The coefficients a, b, c, and d are all unconstrained and not dependent on each other.

$\Rightarrow$ 4 deg of freedom

$\therefore$ $\boxed{\dim(V) = 4}$

b) $V^* = \text{Hom}(V, \mathbb{R})$

maps every element in V to some scalar in $\mathbb{R}$

e.g. if $V = \mathbb{R}^4$

then $V^* = \{ax_1 + bx_2 + cx_3 + dx_4 \mid a, b, c, d \in \mathbb{R}\}$

In order to be a homomorphism where every element is mapped to $\mathbb{R}$, V^* must have the same number of deg of freedom as V, which implies

$$\dim(V) = \dim(V^*)$$

$\therefore$ $\boxed{\dim(V^*) = 4}$

c) Basis for $V = \{ax^3 + bx^2 + cx + d \mid a, b, c, d \in \mathbb{R}\}$

let $f_1(x) = 1$
$f_2(x) = x$
$f_3(x) = x^2$
$f_4(x) = x^3$

then $\{f_1(x), f_2(x), f_3(x), f_4(x)\}$ span all possible 3^{rd} deg polynomials in V

And $c_3x^3 + c_2x^2 + c_1x + c_0 = 0$ for all x iff

$c_3 = c_2 = c_1 = c_0 = 0$

thus, $f_1(x), f_2(x), f_3(x)$, and $f_4(x)$ are linearly independent

5. cont

$\therefore$ $\boxed{\text{basis for } V = \{x^3, x^2, x, 1\}}$

d) Find a basis for $V^* = \text{Hom}(V, \mathbb{R})$

let $F(x) = a_3 f_4(x) + a_2 f_3(x) + a_1 f_2(x) + a_0 f_1(x)$

$= a_3 x^3 + a_2 x^2 + a_1 x + a_0$

(encompasses all polynomials in V)

- In order to map $V \to \mathbb{R}$, want a function $g(F(x)) \in V$ which extracts just the scalar coefficient terms $a_3, a_2, a_1,$ and a_0 in $F(x) \in V$
- Since $\dim(V^*) = 4$, need 4 such functions $g_1(F(x)), \ldots, g_4(F(x))$ in order to form a basis for V^* (one for each coefficient)

thus, define:

1. $g_1 : V \to \mathbb{R}$
   $g_1(F(x)) = a_0$
2. $g_2 : V \to \mathbb{R}$
   $g_2(F(x)) = a_1$
3. $g_3 : V \to \mathbb{R}$
   $g_3(F(x)) = a_2$
4. $g_4 : V \to \mathbb{R}$
   $g_4(F(x)) = a_3$

for all $F(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \in V$

$\{g_1, \ldots, g_4\}$ are linearly independent and span all V^*

$\therefore$ $\boxed{\text{basis for } V^* = \{g_1, g_2, g_3, g_4\}}$

CALCULUS REVIEW (Lectures 8/30, 9/1)

1. Claim: $1 + 2 + 3 + 4 + \cdots = -1/12$

let $a_n = \{1, 2, 3, \ldots\} = \{\mathbb{N}\}$

then $\lim_{n\to\infty} a_n = \infty$

thus, $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges by the divergence test (an infinite series a_n can only converge to a finite number iff $\lim_{n\to\infty} a_n = 0$)

$\therefore \sum_{n=1}^{\infty} a_n = 1 + 2 + 3 + 4 + \cdots \neq -1/12$ since it must diverge

2. a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

because form of $\sum_{n=1}^{\infty} \frac{1}{n^p}$, which always converges for $p > 1$ (by the Integral test)

b) $\sum_{n=1}^{\infty} \frac{1}{n!}$

Apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad \begin{cases} L < 1 \Rightarrow \text{abs. converge} \\ L > 1 \Rightarrow \text{diverge} \\ L = 1 \Rightarrow \text{inconclusive} \end{cases}$$

let $a_n = \frac{1}{n!} \Rightarrow$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \left| \frac{n!}{(n+1)!} \right| = \left| \frac{1}{n+1} \right|$$

then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{1}{n+1} \right| = 0 < 1$$

Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$ which is less than 1, then $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges absolutely by the ratio test (which implies regular convergence)

(2.) c) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$

cont.

Apply the Ratio Test again!

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ where $\begin{cases} L < 1 \Rightarrow \text{absolute convergence} \\ L > 1 \Rightarrow \text{divergence} \\ L = 1 \Rightarrow \text{inconclusive} \end{cases}$

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2 n!}{(n+1)!\, n^2} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{(n+1)\, n^2} \right|$$

$1/\infty$

$$= \lim_{n\to\infty} \left| \frac{n+1}{n^2} \right| = 0 < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{n^2}{n!}$ converges absolutely by the ratio test (and thus converges in general)

(3.) Let a_n = sequence of ± 1, i.e. $a_n = (-1)^n$

Show $\sum_{n=1}^{\infty} \frac{a_n}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ is convergent

let $b_n = (-1)^n / 2^n$

From the ratio test,

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{2^{n+1}}}{\frac{(-1)^n}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} 2^n}{(-1)^n 2^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| (-1)^{n+1-n} 2^{n-(n+1)} \right| = \lim_{n\to\infty} \left| (-1)(2^{-1}) \right|$$

$$= \lim_{n\to\infty} \left| -\frac{1}{2} \right| = \lim_{n\to\infty} \left(\frac{1}{2} \right) = \frac{1}{2} = L$$

Ratio test says that if $\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = L < 1$, then the series b_n converges absolutely

3. cont.

If the absolute value of a series converges, the original series must also converge. That is,

since $\sum_{n=1}^{\infty} \left| \frac{a_n}{2^n} \right| = \sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, then

$\sum_{n=1}^{\infty} \frac{a_n}{2^n}$ must also converge by the ratio test.

(where $a_n = \{-1, 1, -1, 1, \dots\} = (-1)^n \Rightarrow |a_n| = \{1, 1, \dots\}$)

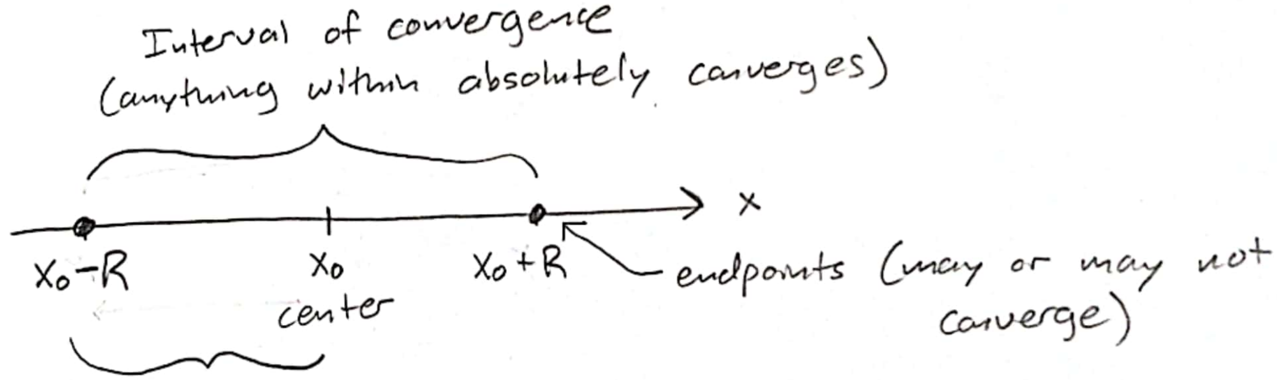
4. a) <u>What is radius of convergence?</u>

Define a function of the form

$$f(x) = \sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

(aka a Taylor Series)

- Then the interval of convergence is the range of specific x values for which the series converges, centered about a point
- The <u>radius</u> of convergence is the width of this interval:



Radius of convergence is R s.t. $f(x)$ converges on the interval $x_0 \pm R$

b) Does $\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n$ hold for all $x \neq 1$?

↳ Check radius of convergence

(4.) Apply ratio test:

cont. $f(x) = \sum_{n=0}^{\infty} x^n \Rightarrow$ let $a_n = x^n$

then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty} |x|$

For convergence, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

thus, $\sum_{n=0}^{\infty} x^n$ converges only for $|x| < 1$

$\Rightarrow -1 < x < 1$

$\Rightarrow$ The radius of convergence is $R = 1$

$\therefore \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$ is <u>not</u> true for all $x \neq 1$, but rather only holds true on the interval $-1 < x < 1$

(we can see in this case that the endpoints are non-inclusive b/c $\sum_{n=0}^{\infty} (-1)^n = 0 \neq \frac{1}{1-(-1)} = \frac{1}{2}$, and $\sum_{n=0}^{\infty} 1^n$ clearly diverges)

(5.) <u>Given:</u> $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

[Harmonic series]

Does $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converge?

<u>Yes</u>, because it is an alternating series of the form $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ (where $a_n = \frac{1}{n}$) that satisfies the following conditions:

next page $\rightarrow$

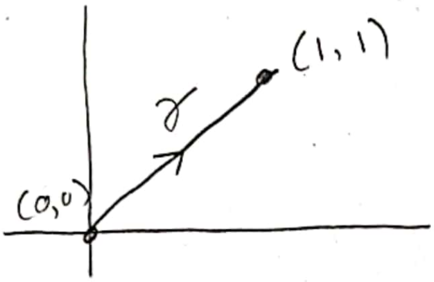
(5.) cont.

1. $a_n = \frac{1}{n}$ is <u>decreasing</u> for all $n > 0$

2. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ (passes the divergence test)

Thus, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ must converge by the <u>alternating series test</u>

(6.) a) Given: γ = straight line from $(0,0)$ to $(1,1)$



Compute $\int_\gamma 2\,dx + 3\,dy$

• Parametrize the line segment: let $\begin{cases} x(t) = t \\ y(t) = t \end{cases}$

for $0 \le t \le 1$

then $dx = dt$ and $dy = dt$

thus, $\int_\gamma 2\,dx + 3\,dy = \int_0^1 2\,dt + 3\,dt = \int_0^1 5\,dt = 5t \Big|_0^1$

$= \boxed{5}$

b) What if γ is replaced by a curved line from $(0,0)$ to $(1,1)$?

→ <u>No</u>, the previous result would not change, because the vector field $\underline{F}(x,y) = (2, 3)$ is <u>conservative</u> and thus path independent (all that matters is the endpoints of the curve)

(6.) cont.

We can verify that the field which the curve γ travels through is conservative by looking at its partial derivatives:

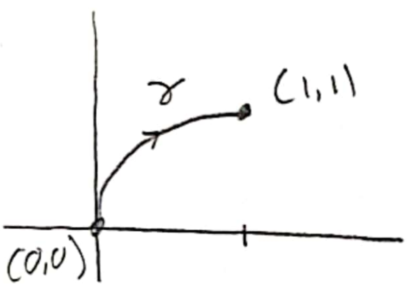
$$\underline{F} = P\hat{i} + Q\hat{j} = 2\hat{i} + 3\hat{j}$$

$$\Rightarrow \quad \frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}$$

(Continuous, first-order on a domain D).

Consequently, $\operatorname{curl}(\underline{F}) = \underline{0}$

To check, try solving line integral using a curved, quarter-circle path:



Parametrize:

$x(t) = 1 - \cos t \quad \Rightarrow \quad dx = \sin t \, dt$

$y(t) = \sin t \qquad\qquad dy = \cos t \, dt$

for $0 \le t \le \frac{\pi}{2}$

then

$$\int_\gamma 2dx + 3dy = \int_0^{\pi/2} 2\sin t \, dt + 3\cos t \, dt$$

$$= \int_0^{\pi/2} (2\sin t + 3\cos t)\, dt$$

$$= (-2\cos t + 3\sin t)\Big|_0^{\pi/2}$$

$$= \cancel{-2\cos\left(\tfrac{\pi}{2}\right)} + \underset{3}{3\sin\left(\tfrac{\pi}{2}\right)} + \overset{2}{2\cos(0)} - \cancel{3\sin(0)}$$

$$= \boxed{5}$$ ✓ same result as before