1. Determine whether the following sequence converges or not

(a) $\lim_{x \to \infty} x^n \cdot e^{-x} = 0$ $\qquad \frac{x^n}{e^x}$

By Taylor expansion, $e^x \geq \frac{x^N}{N!}$ for any N $\quad \therefore \frac{x^n}{e^x} \leq \frac{x^n}{x^N} \cdot N!$

choose $N > n$, then as $x \to \infty$, $\frac{x^n}{x^N} \to 0$.

(b) $\lim_{n \to \infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = \lim_{n \to \infty} e^{\frac{1}{n}\log\left(\frac{1}{n}\right)} = \lim_{n \to \infty} e^{-\frac{\log n}{n}} = e^{-0} = 1$

$\because \lim_{n \to \infty} \frac{\log n}{n} \underset{\uparrow \text{ L'hoptial}}{=} \lim_{n \to \infty} \frac{\frac{1}{n}}{1} = 0$

(c) $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = \lim_{u \to \infty} \frac{\ln(u^2)}{u} = \lim_{u \to \infty} \frac{2 \cdot \ln u}{u} = 0$

(d) $\sum_n \frac{1}{n \cdot \log n}$, by integral test, it diverges or converges the same way as $\int_R^\infty \frac{1}{x} \frac{1}{\log x} dx = \int_{R'}^\infty \frac{1}{\log x} d\log x$

$= \int_{R'}^\infty \frac{1}{u} du = \infty$

so it diverges

(e) $\sum_n \frac{e^n}{\sqrt{n!}}$, consider the ratio test

$\frac{a_{n+1}}{a_n} = \frac{e^{n+1}/\sqrt{(n+1)!}}{e^n/\sqrt{n!}} = \frac{e}{\sqrt{n+1}} \to 0$

as $n \to \infty$ $\quad \therefore$ the series converges.

2. (a) $\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$, $\mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$,

$|z|^2 = z \cdot \bar{z}$

(b) $|z/\bar{z}| = 1$ for any $z \neq 0$

(c) $z = (\frac{1}{2}) \cdot e^{i\pi/3}$, $\bar{z} = (\frac{1}{2}) e^{-i\pi/3}$

$1/\bar{z} = 2 \cdot e^{i\pi/3}$

(d) $z^3 = e^{i\pi/2} \Rightarrow z = e^{i\pi/6 + \frac{2\pi i}{3} k}$ $k = 0, 1, 2.$

(e) $\mathrm{Re}(e^{iz}) = \frac{1}{2}(e^{iz} + e^{-i\bar{z}})$

$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$

$$\mathrm{Re}(e^{iz}) = \cos(z)$$
$$\Leftrightarrow \quad e^{-i\bar{z}} = e^{-iz}$$
$$\Leftrightarrow \quad -i\bar{z} = -iz + 2\pi i \cdot n$$
$$\Leftrightarrow \quad z - \bar{z} = 2\pi n \quad \text{for some } n \in \mathbb{Z}$$

but $z - \bar{z}$ is purely imaginary, $2\pi n$ is pure real, hence this is only possible if $z = \bar{z}$, i.e. $z \in \mathbb{R}$

Here $z = 1+2i$ is not real

$\therefore$ $\mathrm{Re}(e^{iz}) \neq \cos(z)$ for $z = 1+2i$

3. (a)

$$f(z) = \frac{z}{(z+1)^2} = z\left(1 - z + z^2 + \cdots\right)^2$$
$$= z\left(1 - 2z + \cdots\right)$$
$$= z - 2z^2 + \cdots$$

(b) $f(z) = \dfrac{z}{(z+1)^2}$ set $u = z+1$, then $z = u-1$

$$= \frac{u-1}{u^2} = -\frac{1}{u^2} + \frac{1}{u} = -\frac{1}{(z+1)^2} + \frac{1}{z+1}$$

4 $$\oint_{|z|=2} \frac{1}{z(z+1)(z+4)(z+5)}\, dz$$

$$= 2\pi i \cdot \left(\operatorname{Res}_{z=0} + \operatorname{Res}_{z=-1}\right) \frac{1}{z(z+1)(z+4)(z+5)}$$

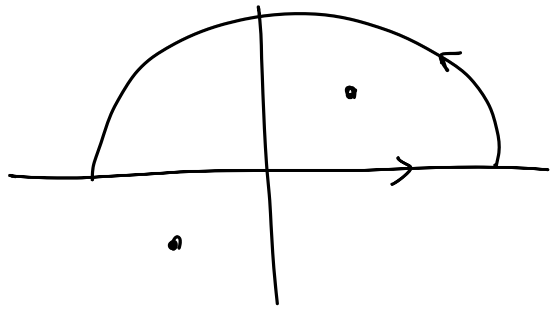
$$= 2\pi i \cdot \left(\frac{1}{(0+1)(0+4)(0+5)} + \frac{1}{(-1)(-1+4)(-1+5)}\right)$$

$$= 2\pi i \left(\frac{1}{20} - \frac{1}{12}\right)$$

#5 $$I = \int_{-\infty}^{+\infty} \frac{1}{1+ix^2}\, dx$$

the pole of $\dfrac{1}{1+iz^2}$ is at

$1 + iz^2 = 0 \iff z^2 = +i \iff z = \pm e^{i\frac{\pi}{4}}$



By the same argument as in class

$$I = \oint_{\substack{\text{size } R \\ R \gg 1}} \frac{1}{1+iz^2}\, dz = 2\pi i \cdot \operatorname{Res}_{z=e^{i\frac{\pi}{4}}} \frac{1}{1+iz^2}$$

$$= 2\pi i \cdot \frac{1}{2i \cdot e^{i\frac{\pi}{4}}} = \pi \cdot e^{-i\frac{\pi}{4}}$$

Alternative method :

let $u = \sqrt{i} \cdot x = e^{i\frac{\pi}{4}} \cdot x$

then as x goes along $\mathbb{R}$, u goes along

$e^{i\frac{\pi}{4}} \cdot \mathbb{R}$

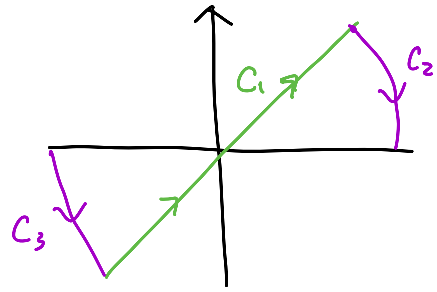
$$I = \int_{u \in e^{i\frac{\pi}{4}} \cdot \mathbb{R}} \frac{1}{1+u^2} \cdot e^{-i\frac{\pi}{4}}\, du$$

$$= e^{-i\frac{\pi}{4}} \cdot \int_{u \in \mathbb{R}} \frac{1}{1+u^2}\, du$$

$$= e^{-i\frac{\pi}{4}} \cdot \pi$$

⌈To be rigourous, we did

$$\lim_{R\to\infty} \int_{C_1} \frac{1}{1+u^2}\,du$$



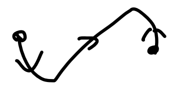
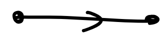
$$= \lim_{R\to\infty} \int_{C_1} \frac{1}{1+u^2}\,du + \lim_{R\to\infty} \int_{C_2} \frac{1}{1+u^2}\,du + \lim_{R\to\infty} \int_{C_3} \frac{1}{1+u^2}\,du$$

($\because$ the 2 added terms $= 0$)

$$= \lim_{R\to\infty} \int_{C_1+C_2+C_3} \frac{1}{1+u^2}\,du$$

$$= \lim_{R\to\infty} \int_{u=-R}^{R} \frac{1}{1+u^2}\,du$$

$$= \int_{-\infty}^{+\infty} \frac{1}{1+u^2}\,du.$$

we deform contour from  to  without changing the end points, and without touching the singularity at $u = \pm i$