9. This exercise is concerned with vector spaces that need not be finite-dimensional; most of its parts (but not all) depend on the sort of transfinite reasoning that is needed to prove that every vector space has a basis (cf. § 7, Ex. 11).

(a) Suppose that f and g are scalar-valued functions defined on a set $\mathfrak{X}$; if α and β are scalars write $h = \alpha f + \beta g$ for the function defined by $h(x) = \alpha f(x) + \beta g(x)$ for all x in $\mathfrak{X}$. The set of all such functions is a vector space with respect to this definition of the linear operations, and the same is true of the set of all finitely non-zero functions. (A function f on $\mathfrak{X}$ is *finitely non-zero* if the set of those elements x of $\mathfrak{X}$ for which $f(x) \neq 0$ is finite.)

(b) Every vector space is isomorphic to the set of all finitely non-zero functions on some set.

(c) If $\mathfrak{V}$ is a vector space with basis $\mathfrak{X}$, and if f is a scalar-valued function defined on the set $\mathfrak{X}$, then there exists a unique linear functional y on $\mathfrak{V}$ such that $[x, y] = f(x)$ for all x in $\mathfrak{X}$.

(d) Use (a), (b), and (c) to conclude that every vector space $\mathfrak{V}$ is isomorphic to a subspace of $\mathfrak{V}'$.

(e) Which vector spaces are isomorphic to their own duals?

(f) If $\mathfrak{Y}$ is a linearly independent subset of a vector space $\mathfrak{V}$, then there exists a basis of $\mathfrak{V}$ containing $\mathfrak{Y}$. (Compare this result with the theorem of § 7.)

(g) If $\mathfrak{X}$ is a set and if y is an element of $\mathfrak{X}$, write f_y for the scalar-valued function defined on $\mathfrak{X}$ by writing $f_y(x) = 1$ or 0 according as $x = y$ or $x \neq y$. Let $\mathfrak{Y}$ be the set of all functions f_y together with the function g defined by $g(x) = 1$ for all x in $\mathfrak{X}$. Prove that if $\mathfrak{X}$ is infinite, then $\mathfrak{Y}$ is a linearly independent subset of the vector space of all scalar-valued functions on $\mathfrak{X}$.

(h) The natural correspondence from $\mathfrak{V}$ to $\mathfrak{V}''$ is defined for all vector spaces (not only for the finite-dimensional ones); if x_0 is in $\mathfrak{V}$, define the corresponding element z_0 of $\mathfrak{V}''$ by writing $z_0(y) = [x_0, y]$ for all y in $\mathfrak{V}'$. Prove that if $\mathfrak{V}$ is reflexive (i.e., if every z_0 in $\mathfrak{V}''$ can be obtained in this manner by a suitable choice of x_0), then $\mathfrak{V}$ is finite-dimensional. (Hint: represent $\mathfrak{V}'$ as the set of all scalar-valued functions on some set, and then use (g), (f), and (c) to construct an element of $\mathfrak{V}''$ that is not induced by an element of $\mathfrak{V}$.)

Warning: the assertion that a vector space is reflexive if and only if it is finite-dimensional would shock most of the experts in the subject. The reason is that the customary and fruitful generalization of the concept of reflexivity to infinite-dimensional spaces is not the simple-minded one given in (h).

§ 18. Direct sums

We shall study several important general methods of making new vector spaces out of old ones; in this section we begin by studying the easiest one.

DEFINITION. If $\mathfrak{U}$ and $\mathfrak{V}$ are vector spaces (over the same field), their *direct sum* is the vector space $\mathfrak{W}$ (denoted by $\mathfrak{U} \oplus \mathfrak{V}$) whose elements are all the ordered pairs $\langle x, y\rangle$ with x in $\mathfrak{U}$ and y in $\mathfrak{V}$, with the linear operations defined by

$$\alpha_1\langle x_1, y_1\rangle + \alpha_2\langle x_2, y_2\rangle = \langle \alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2\rangle.$$

We observe that the formation of the direct sum is analogous to the way in which the plane is constructed from its two coordinate axes.

We proceed to investigate the relation of this notion to some of our earlier ones.

The set of all vectors (in $\mathcal{W}$) of the form $\langle x, 0\rangle$ is a subspace of $\mathcal{W}$; the correspondence $\langle x, 0\rangle \rightleftarrows x$ shows that this subspace is isomorphic to $\mathcal{U}$. It is convenient, once more, to indulge in a logical inaccuracy and, identifying x and $\langle x, 0\rangle$, to speak of $\mathcal{U}$ as a subspace of $\mathcal{W}$. Similarly, of course, the vectors y of $\mathcal{V}$ may be identified with the vectors of the form $\langle 0, y\rangle$ in $\mathcal{W}$, and we may consider $\mathcal{V}$ as a subspace of $\mathcal{W}$. This terminology is, to be sure, not quite exact, but the logical difficulty is much easier to get around here than it was in the case of the second dual space. We could have defined the direct sum of $\mathcal{U}$ and $\mathcal{V}$ (at least in the case in which $\mathcal{U}$ and $\mathcal{V}$ have no non-zero vectors in common) as the set consisting of all x's in $\mathcal{U}$, all y's in $\mathcal{V}$, and all those pairs $\langle x, y\rangle$ for which $x \neq 0$ and $y \neq 0$. This definition yields a theory analogous in every detail to the one we shall develop, but it makes it a nuisance to prove theorems because of the case distinctions it necessitates. It is clear, however, that from the point of view of this definition $\mathcal{U}$ is actually a subset of $\mathcal{U} \oplus \mathcal{V}$. In this sense then, or in the isomorphism sense of the definition we did adopt, we raise the question: what is the relation between $\mathcal{U}$ and $\mathcal{V}$ when we consider these spaces as subspaces of the big space $\mathcal{W}$?

THEOREM. *If $\mathcal{U}$ and $\mathcal{V}$ are subspaces of a vector space $\mathcal{W}$, then the following three conditions are equivalent.*

(1) $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$.

(2) $\mathcal{U} \cap \mathcal{V} = \mathcal{O}$ *and* $\mathcal{U} + \mathcal{V} = \mathcal{W}$ (*i.e.,* $\mathcal{U}$ *and* $\mathcal{V}$ *are complements of each other*).

(3) *Every vector z in $\mathcal{W}$ may be written in the form $z = x + y$, with x in $\mathcal{U}$ and y in $\mathcal{V}$, in one and only one way.*

PROOF. We shall prove the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). We assume that $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$. If $z = \langle x, y\rangle$ lies in both $\mathcal{U}$ and $\mathcal{V}$, then $x = y = 0$, so that $z = 0$; this proves that $\mathcal{U} \cap \mathcal{V} = \mathcal{O}$. Since the representation $z = \langle x, 0\rangle + \langle 0, y\rangle$ is valid for every z, it follows also that $\mathcal{U} + \mathcal{V} = \mathcal{W}$.

(2) $\Rightarrow$ (3). If we assume (2), so that, in particular, $\mathcal{U} + \mathcal{V} = \mathcal{W}$, then it is clear that every z in $\mathcal{W}$ has the desired representation, $z = x + y$. To prove uniqueness, we assume that $z = x_1 + y_1$ and $z = x_2 + y_2$, with x_1 and x_2 in $\mathcal{U}$ and y_1 and y_2 in $\mathcal{V}$. Since $x_1 + y_1 = x_2 + y_2$, it follows that $x_1 - x_2 = y_2 - y_1$. Since the left member of this last equation is in $\mathcal{U}$ and the right member is in $\mathcal{V}$, the disjointness of $\mathcal{U}$ and $\mathcal{V}$ implies that $x_1 = x_2$ and $y_1 = y_2$.

(3) $\Rightarrow$ (1). This implication is practically indistinguishable from the definition of direct sum. If we form the direct sum $\mathcal{U} \oplus \mathcal{V}$, and then

identify $\langle x, 0\rangle$ and $\langle 0, y\rangle$ with x and y respectively, we are committed to identifying the sum $\langle x, y\rangle = \langle x, 0\rangle + \langle 0, y\rangle$ with what we are assuming to be the general element $z = x + y$ of $\mathcal{W}$; from the hypothesis that the representation of z in the form $x + y$ is unique we conclude that the correspondence between $\langle x, 0\rangle$ and x (and also between $\langle 0, y\rangle$ and y) is one-to-one.

If two subspaces $\mathcal{U}$ and $\mathcal{V}$ in a vector space $\mathcal{W}$ are disjoint and span $\mathcal{W}$ (that is, if they satisfy (2)), it is usual to say that $\mathcal{W}$ is the *internal direct sum* of $\mathcal{U}$ and $\mathcal{V}$; symbolically, as before, $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$. If we want to emphasize the distinction between this concept and the one defined before, we describe the earlier one by saying that $\mathcal{W}$ is the *external direct sum* of $\mathcal{U}$ and $\mathcal{V}$. In view of the natural isomorphisms discussed above, and, especially, in view of the preceding theorem, the distinction is more pedantic than conceptual. In accordance with our identification convention, we shall usually ignore it.

§ 19. Dimension of a direct sum

What can be said about the dimension of a direct sum? If $\mathcal{U}$ is n-dimensional, $\mathcal{V}$ is m-dimensional, and $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$, what is the dimension of $\mathcal{W}$? This question is easy to answer.

THEOREM 1. *The dimension of a direct sum is the sum of the dimensions of its summands.*

PROOF. We assert that if $\{x_1, \cdots, x_n\}$ is a basis in $\mathcal{U}$, and if $\{y_1, \cdots, y_m\}$ is a basis in $\mathcal{V}$, then the set $\{x_1, \cdots, x_n, y_1, \cdots, y_m\}$ (or, more precisely, the set $\{\langle x_1, 0\rangle, \cdots, \langle x_n, 0\rangle, \langle 0, y_1\rangle, \cdots, \langle 0, y_m\rangle\}$) is a basis in $\mathcal{W}$. The easiest proof of this assertion is to use the implication (1) $\Rightarrow$ (3) from the theorem of the preceding section. Since every z in $\mathcal{W}$ may be written in the form $z = x + y$, where x is a linear combination of $x_1, \cdots, x_n$ and y is a linear combination of $y_1, \cdots, y_m$, it follows that our set does indeed span $\mathcal{W}$. To show that the set is also linearly independent, suppose that

$$\alpha_1 x_1 + \cdots + \alpha_n x_n + \beta_1 y_1 + \cdots + \beta_m y_m = 0.$$

The uniqueness of the representation of 0 in the form $x + y$ implies that

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = \beta_1 y_1 + \cdots + \beta_m y_m = 0,$$

and hence the linear independence of the x's and of the y's implies that

$$\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_m = 0.$$

THEOREM 2. *If $\mathcal{W}$ is any $(n+m)$-dimensional vector space, and if $\mathcal{U}$ is any n-dimensional subspace of $\mathcal{W}$, then there exists an m-dimensional subspace $\mathcal{V}$ in $\mathcal{W}$ such that $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$.*

PROOF. Let $\{x_1, \cdots, x_n\}$ be any basis in $\mathcal{U}$; by the theorem of § 7 we may find a set $\{y_1, \cdots, y_m\}$ of vectors in $\mathcal{W}$ with the property that $\{x_1, \cdots, x_n, y_1, \cdots, y_m\}$ is a basis in $\mathcal{W}$. Let $\mathcal{V}$ be the subspace spanned by $y_1, \cdots, y_m$; we omit the verification that $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$.

Theorem 2 says that every subspace of a finite-dimensional vector space has a complement.

§ 20. Dual of a direct sum

In most of what follows we shall view the notion of direct sum as defined for subspaces of a vector space $\mathcal{V}$; this avoids the fuss with the identification convention of § 18, and it turns out, incidentally, to be the more useful concept for our later work. We conclude, for the present, our study of direct sums, by observing the simple relation connecting dual spaces, annihilators, and direct sums. To emphasize our present view of direct summation, we return to the letters of our earlier notation.

THEOREM. *If $\mathfrak{M}$ and $\mathfrak{N}$ are subspaces of a vector space $\mathcal{V}$, and if $\mathcal{V} = \mathfrak{M} \oplus \mathfrak{N}$, then $\mathfrak{M}'$ is isomorphic to $\mathfrak{N}^0$ and $\mathfrak{N}'$ to $\mathfrak{M}^0$, and $\mathcal{V}' = \mathfrak{M}^0 \oplus \mathfrak{N}^0$.*

PROOF. To simplify the notation we shall use, throughout this proof, x, x', and x^0 for elements of $\mathfrak{M}$, $\mathfrak{M}'$, and $\mathfrak{M}^0$, respectively, and we reserve, similarly, the letters y for $\mathfrak{N}$ and z for $\mathcal{V}$. (This notation is not meant to suggest that there is any particular relation between, say, the vectors x in $\mathfrak{M}$ and the vectors x' in $\mathfrak{M}'$.)

If z' belongs to both $\mathfrak{M}^0$ and $\mathfrak{N}^0$, i.e., if $z'(x) = z'(y) = 0$ for all x and y, then $z'(z) = z'(x + y) = 0$ for all z; this implies that $\mathfrak{M}^0$ and $\mathfrak{N}^0$ are disjoint. If, moreover, z' is any vector in $\mathcal{V}'$, and if $z = x + y$, we write $x^0(z) = z'(y)$ and $y^0(z) = z'(x)$. It is easy to see that the functions x^0 and y^0 thus defined are linear functionals on $\mathcal{V}$ (i.e., elements of $\mathcal{V}'$) belonging to $\mathfrak{M}^0$ and $\mathfrak{N}^0$ respectively; since $z' = x^0 + y^0$, it follows that $\mathcal{V}'$ is indeed the direct sum of $\mathfrak{M}^0$ and $\mathfrak{N}^0$.

To establish the asserted isomorphisms, we make correspond to every x^0 a y' in $\mathfrak{N}'$ defined by $y'(y) = x^0(y)$. We leave to the reader the routine verification that the correspondence $x^0 \to y'$ is linear and one-to-one, and therefore an isomorphism between $\mathfrak{M}^0$ and $\mathfrak{N}'$; the corresponding result for $\mathfrak{N}^0$ and $\mathfrak{M}'$ follows from symmetry by interchanging x and y. (Observe that for finite-dimensional vector spaces the mere existence of an isomorphism between, say, $\mathfrak{M}^0$ and $\mathfrak{N}'$ is trivial from a dimension argument; *indeed, the* dimensions of both $\mathfrak{M}^0$ and $\mathfrak{N}'$ are equal to the dimension of $\mathfrak{N}$.)

We remark, concerning our entire presentation of the theory of direct sums, that there is nothing magic about the number two; we could have defined the direct sum of any finite number of vector spaces, and we could have proved the obvious analogues of all the theorems of the last three sections, with only the notation becoming more complicated. We serve warning that we shall use this remark later and treat the theorems it implies as if we had proved them.

EXERCISES

1. Suppose that x, y, u, and v are vectors in $\mathbb{C}^4$; let $\mathfrak{M}$ and $\mathfrak{N}$ be the subspaces of $\mathbb{C}^4$ spanned by $\{x, y\}$ *and* $\{u, v\}$ respectively. In which of the following cases is it true that $\mathbb{C}^4 = \mathfrak{M} \oplus \mathfrak{N}$?
(a) $x = (1, 1, 0, 0)$, $y = (1, 0, 1, 0)$
$u = (0, 1, 0, 1)$, $v = (0, 0, 1, 1)$.
(b) $x = (-1, 1, 1, 0)$, $y = (0, 1, -1, 1)$
$u = (1, 0, 0, 0)$, $v = (0, 0, 0, 1)$.
(c) $x = (1, 0, 0, 1)$, $y = (0, 1, 1, 0)$
$u = (1, 0, 1, 0)$, $v = (0, 1, 0, 1)$.

2. If $\mathfrak{M}$ is the subspace consisting of all those vectors $(\xi_1, \cdots, \xi_n, \xi_{n+1}, \cdots, \xi_{2n})$ in $\mathbb{C}^{2n}$ for which $\xi_1 = \cdots = \xi_n = 0$, and if $\mathfrak{N}$ is the subspace of all those vectors for which $\xi_j = \xi_{n+j}$, $j = 1, \cdots, n$, then $\mathbb{C}^{2n} = \mathfrak{M} \oplus \mathfrak{N}$.

3. Construct three subspaces $\mathfrak{M}$, $\mathfrak{N}_1$, and $\mathfrak{N}_2$ of a vector space $\mathfrak{V}$ so that $\mathfrak{M} \oplus \mathfrak{N}_1 = \mathfrak{M} \oplus \mathfrak{N}_2 = \mathfrak{V}$ but $\mathfrak{N}_1 \neq \mathfrak{N}_2$. (Note that this means that there is no cancellation law for direct sums.) What is *the geometric* picture corresponding to this situation?

4. (a) If $\mathfrak{U}$, $\mathfrak{V}$, and $\mathfrak{W}$ are vector spaces, what is the relation between $\mathfrak{U} \oplus (\mathfrak{V} \oplus \mathfrak{W})$ and $(\mathfrak{U} \oplus \mathfrak{V}) \oplus \mathfrak{W}$ (i.e., in what sense is the formation of direct sums an associative operation)?
(b) In what sense is the formation of direct sums commutative?

5. (a) Three subspaces $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{N}$ of a vector space $\mathfrak{V}$ are called *independent* if each one is disjoint from the sum of the other two. Prove that a necessary and sufficient condition for $\mathfrak{V} = \mathfrak{L} \oplus (\mathfrak{M} \oplus \mathfrak{N})$ (and also for $\mathfrak{V} = (\mathfrak{L} \oplus \mathfrak{M}) \oplus \mathfrak{N}$) is that $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{N}$ be independent and that $\mathfrak{V} = \mathfrak{L} + \mathfrak{M} + \mathfrak{N}$. (The subspace $\mathfrak{L} + \mathfrak{M} + \mathfrak{N}$ is the set of all vectors of the form $x + y + z$, with x in $\mathfrak{L}$, y in $\mathfrak{M}$, and z in $\mathfrak{N}$.)
(b) Give an example of three subspaces *of* a vector space $\mathfrak{V}$, such that the sum of all *three is* $\mathfrak{V}$, such that every two of the three are disjoint, but such that the three are not independent.
(c) Suppose that x, y, and z are elements of a vector space and *that* $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{N}$ are the subspaces spanned by x, y, and z, respectively. Prove that the vectors x, y, and z are linearly independent if and only if the subspaces $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{N}$ are independent.
(d) Prove that three finite-dimensional subspaces are independent if and only if the sum of their dimensions is equal to the dimension of their sum.
(e) Generalize the results (a)–(d) from three subspaces to any finite number.

§ 21. Quotient spaces

We know already that if $\mathfrak{M}$ is a subspace of a vector space $\mathfrak{V}$, then there are, usually, many other *subspaces* $\mathfrak{N}$ in $\mathfrak{V}$ such that $\mathfrak{M} \oplus \mathfrak{N} = \mathfrak{V}$. There is no natural way of choosing one from among the wealth of complements of $\mathfrak{M}$. There is, however, a natural construction that associates with $\mathfrak{M}$ and $\mathfrak{V}$ a new vector space that, for all practical purposes, plays the role of a complement of $\mathfrak{M}$. The theoretical advantage that the construction has over the formation of an arbitrary complement is precisely its "natural" character, i.e., the fact that it does not depend on choosing a basis, or, for that matter, on choosing anything at all.

In order to understand the construction it is a good idea to keep a picture in mind. Suppose, for instance, that $\mathfrak{V} = \mathfrak{R}^2$ (the real coordinate plane) and that $\mathfrak{M}$ consists of all those vectors (ξ_1, ξ_2) for which $\xi_2 = 0$ (the horizontal axis). Each complement of $\mathfrak{M}$ is a line (other than the horizontal axis) through the origin. Observe that each such complement has the property that it intersects every horizontal line in exactly one point. The idea of the construction we shall describe is to make a vector space out of the set of all horizontal lines.

We begin by using $\mathfrak{M}$ to single out certain subsets of $\mathfrak{V}$. (We are back in the general case now.) If x is an arbitrary vector in $\mathfrak{V}$, we write $x + \mathfrak{M}$ for the set of all sums $x + y$ with y in $\mathfrak{M}$; each set of the form $x + \mathfrak{M}$ is called a *coset* of $\mathfrak{M}$. (In the case of the plane-line example above, the cosets are the horizontal lines.) Note that one and the same coset can arise from two different vectors, i.e., that even if $x \neq y$, it is possible that $x + \mathfrak{M} = y + \mathfrak{M}$. It makes good sense, just the same, to speak of a coset, say $\mathfrak{H}$, of $\mathfrak{M}$, *without specifying* which element (or elements) $\mathfrak{H}$ comes from; to say that $\mathfrak{H}$ is a coset (of $\mathfrak{M}$) means simply that there is at least one x such that $\mathfrak{H} = x + \mathfrak{M}$.

If $\mathfrak{H}$ and $\mathfrak{K}$ are cosets (of $\mathfrak{M}$), we write $\mathfrak{H} + \mathfrak{K}$ for the set of all sums $u + v$ with u in $\mathfrak{H}$ and v in $\mathfrak{K}$; we assert that $\mathfrak{H} + \mathfrak{K}$ is also a coset of $\mathfrak{M}$. Indeed, if $\mathfrak{H} = x + \mathfrak{M}$ and $\mathfrak{K} = y + \mathfrak{M}$, then every element of $\mathfrak{H} + \mathfrak{K}$ belongs to the coset $(x + y) + \mathfrak{M}$ (note that $\mathfrak{M} + \mathfrak{M} = \mathfrak{M}$), and, conversely, every element of $(x + y) + \mathfrak{M}$ is in $\mathfrak{H} + \mathfrak{K}$. (If, for instance, z is in $\mathfrak{M}$, then $(x + y) + z = (x + z) + (y + 0)$.) In other words, $\mathfrak{H} + \mathfrak{K} = (x + y) + \mathfrak{M}$, so that $\mathfrak{H} + \mathfrak{K}$ is a coset, as asserted. We leave to the reader the verification that coset addition is commutative and associative. The coset $\mathfrak{M}$ (i.e., $0 + \mathfrak{M}$) is such that $\mathfrak{H} + \mathfrak{M} = \mathfrak{H}$ for every coset $\mathfrak{H}$, and, moreover, $\mathfrak{M}$ is the only coset *with this property*. (If $(x + \mathfrak{M}) + (y + \mathfrak{M}) = x + \mathfrak{M}$, then $x + \mathfrak{M}$ contains $x + y$, so that $x + y = x + u$ for some u in $\mathfrak{M}$; this implies that y is in $\mathfrak{M}$, and hence that $y + \mathfrak{M} = \mathfrak{M}$.) If $\mathfrak{H}$ is a coset, then the set consisting of all the vectors $-u$, with u in $\mathfrak{H}$, is itself a coset, which we shall denote by $-\mathcal{H}$. The coset $-\mathcal{H}$ is such that $\mathcal{H} + (-\mathcal{H}) = \mathfrak{M}$, and, moreover, $-\mathcal{H}$ is the only coset with this property. To sum up: the addition of cosets satisfies the axioms (A) of § 2.

If $\mathcal{H}$ is a coset and if α is a scalar, we write $\alpha\mathcal{H}$ for the set consisting of all the vectors αu with u in $\mathcal{H}$ in case $\alpha \neq 0$; the coset $0 \cdot \mathcal{H}$ is defined to be $\mathfrak{M}$. A simple verification shows that this concept of multiplication satisfies the axioms (B) and (C) of § 2.

The set of all cosets has thus been proved to be a vector space with respect to the linear operations defined above. This vector space is called the *quotient space* of $\mathcal{V}$ modulo $\mathfrak{M}$; it is denoted by $\mathcal{V}/\mathfrak{M}$.

§ 22. Dimension of a quotient space

THEOREM 1. *If $\mathfrak{M}$ and $\mathfrak{N}$ are complementary subspaces of a vector space $\mathcal{V}$, then the correspondence that assigns to each vector y in $\mathfrak{N}$ the coset $y + \mathfrak{M}$ is an isomorphism between $\mathfrak{N}$ and $\mathcal{V}/\mathfrak{M}$.*

PROOF. If y_1 and y_2 are elements of $\mathfrak{N}$ such that $y_1 + \mathfrak{M} = y_2 + \mathfrak{M}$, then, in particular, y_1 belongs to $y_2 + \mathfrak{M}$, so that $y_1 = y_2 + x$ for some x in $\mathfrak{M}$. Since this means that $y_1 - y_2 = x$, and since $\mathfrak{M}$ and $\mathfrak{N}$ are disjoint, it follows that $x = 0$, and hence that $y_1 = y_2$. (Recall that $y_1 - y_2$ belongs to $\mathfrak{N}$ along with y_1 and y_2.) This argument proves that the correspondence we are studying is one-to-one, as far as it goes. To prove that it goes far enough, consider an arbitrary coset of $\mathfrak{M}$, say $z + \mathfrak{M}$. Since $\mathcal{V} = \mathfrak{N} + \mathfrak{M}$, we may write z in the form $y + x$, with x in $\mathfrak{M}$ and y in $\mathfrak{N}$; it follows (since $x + \mathfrak{M} = \mathfrak{M}$) that $z + \mathfrak{M} = y + \mathfrak{M}$. This proves that every coset of $\mathfrak{M}$ can be obtained by using an element of $\mathfrak{N}$ (and not just any old element of $\mathcal{V}$); consequently $y \to y + \mathfrak{M}$ is indeed a one-to-one correspondence between $\mathfrak{N}$ and $\mathcal{V}/\mathfrak{M}$. The linear property of the correspondence is immediate from the definition of the linear operations in $\mathcal{V}/\mathfrak{M}$; indeed, we have

$$(\alpha_1 y_1 + \alpha_2 y_2) + \mathfrak{M} = \alpha_1(y_1 + \mathfrak{M}) + \alpha_2(y_2 + \mathfrak{M}).$$

THEOREM 2. *If $\mathfrak{M}$ is an m-dimensional subspace of an n-dimensional vector space $\mathcal{V}$, then $\mathcal{V}/\mathfrak{M}$ has dimension $n - m$.*

PROOF. Use § 19, Theorem 2 to find a subspace $\mathfrak{N}$ so that $\mathfrak{M} \oplus \mathfrak{N} = \mathcal{V}$. The space $\mathfrak{N}$ has dimension $n - m$ (by § 19, Theorem 1), and it is isomorphic to $\mathcal{V}/\mathfrak{M}$ (by Theorem 1 above).

There are more topics in the theory of quotient spaces that we could discuss (such as their relation to dual spaces and annihilators). Since, however, most such topics are hardly more than exercises, involving the use of techniques already at our disposal, we turn instead to some new and non-obvious ways of manufacturing useful vector spaces.

EXERCISES

1. Consider the quotient spaces obtained by reducing the space $\mathcal{P}$ of polynomials modulo various subspaces. If $\mathfrak{M} = \mathcal{P}_n$, is $\mathcal{P}/\mathfrak{M}$ finite-dimensional? What if $\mathfrak{M}$ is the subspace consisting of all even polynomials? What if $\mathfrak{M}$ is the subspace consisting of all polynomials divisible by x_n (where $x_n(t) = t^n$)?

2. If $\mathcal{S}$ and $\mathcal{J}$ are arbitrary subsets of a vector space (not necessarily cosets of a subspace), there is nothing to stop us from defining $\mathcal{S} + \mathcal{J}$ just as addition was defined for cosets, and, similarly, we may define $\alpha\mathcal{S}$ (where α is a scalar). If the class of all subsets of a vector space is endowed with these "linear operations," which of the axioms of a vector space are satisfied?

3. (a) Suppose that $\mathfrak{M}$ is a subspace of a vector space $\mathcal{V}$. Two vectors x and y of $\mathcal{V}$ are *congruent* modulo $\mathfrak{M}$, in symbols $x \equiv y\ (\mathfrak{M})$, if $x - y$ is in $\mathfrak{M}$. Prove that congruence modulo $\mathfrak{M}$ is an *equivalence relation*, i.e., that it is reflexive ($x \equiv x$), symmetric (if $x \equiv y$, then $y \equiv x$), and transitive (if $x \equiv y$ and $y \equiv z$, then $x \equiv z$).

(b) If α_1 and α_2 are scalars, and if x_1, x_2, y_1, and y_2 are vectors such that $x_1 \equiv y_1\ (\mathfrak{M})$ and $x_2 \equiv y_2\ (\mathfrak{M})$, then $\alpha_1 x_1 + \alpha_2 x_2 \equiv \alpha_1 y_1 + \alpha_2 y_2\ (\mathfrak{M})$.

(c) Congruence modulo $\mathfrak{M}$ splits $\mathcal{V}$ into equivalence classes, i.e., into sets such that two vectors belong to the same set if and only if they are congruent. Prove that a subset of $\mathcal{V}$ is an equivalence class modulo $\mathfrak{M}$ if and only if it is a coset of $\mathfrak{M}$.

4. (a) Suppose that $\mathfrak{M}$ is a subspace of a vector space $\mathcal{V}$. Corresponding to every linear functional y on $\mathcal{V}/\mathfrak{M}$ (i.e., to every element y of $(\mathcal{V}/\mathfrak{M})'$), there is a linear functional z on $\mathcal{V}$ (i.e., an element of $\mathcal{V}'$); the linear functional z is defined by $z(x) = y(x + \mathfrak{M})$. Prove that the correspondence $y \to z$ is an isomorphism between $(\mathcal{V}/\mathfrak{M})'$ and $\mathfrak{M}^0$.

(b) Suppose that $\mathfrak{M}$ is a subspace of a vector space $\mathcal{V}$. Corresponding to every coset $y + \mathfrak{M}^0$ of $\mathfrak{M}^0$ in $\mathcal{V}'$ (i.e., to every element $\mathcal{H}$ of $\mathcal{V}'/\mathfrak{M}^0$), there is a linear functional z on $\mathfrak{M}$ (i.e., an element z of $\mathfrak{M}'$); the linear functional z is defined by $z(x) = y(x)$. Prove that z is unambiguously determined by the coset $\mathcal{H}$ (that is, it does not depend on the particular choice of y), and that the correspondence $\mathcal{H} \to z$ is an isomorphism between $\mathcal{V}'/\mathfrak{M}^0$ and $\mathfrak{M}'$.

5. Given a finite-dimensional vector space $\mathcal{V}$, form the direct sum $\mathcal{W} = \mathcal{V} \oplus \mathcal{V}'$, and prove that the correspondence $\langle x, y\rangle \to \langle y, x\rangle$ is an isomorphism between $\mathcal{W}$ and $\mathcal{W}'$.

§ 23. Bilinear forms

If $\mathcal{U}$ and $\mathcal{V}$ are vector spaces (over the same field), then their direct sum $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$ is another vector space; we propose to study certain functions on $\mathcal{W}$. (For present purposes the original definition of $\mathcal{U} \oplus \mathcal{V}$, via ordered pairs, is the convenient one.) The value of such a function, say w, at an element $\langle x, y\rangle$ of $\mathcal{W}$ will be denoted by $w(x, y)$. The study of linear functions on $\mathcal{W}$ is no longer of much interest to us; the principal facts concerning them were discussed in § 20. The functions we want to consider now are the bilinear ones; they are, by definition, the scalar-valued functions on $\mathcal{W}$ with the property that for each fixed value of either argument they depend linearly on the other argument. More precisely, a scalar-valued function w on $\mathcal{W}$ is a *bilinear form* (or *bilinear functional*) if

$$w(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 w(x_1, y) + \alpha_2 w(x_2, y)$$

and

$$w(x, \alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 w(x, y_1) + \alpha_2 w(x, y_2),$$

identically in the vectors and scalars involved.

In one special situation we have already encountered bilinear functionals. If, namely, $\mathcal{V}$ is the dual space of $\mathcal{U}$, $\mathcal{V} = \mathcal{U}'$, and if we write $w(x, y) = [x, y]$ (see § 14), then w is a bilinear functional on $\mathcal{U} \oplus \mathcal{U}'$. For an example in a more general situation, let $\mathcal{U}$ and $\mathcal{V}$ be arbitrary vector spaces (over the same field, as always), let u and v be elements of $\mathcal{U}'$ and $\mathcal{V}'$ respectively, and write $w(x, y) = u(x)v(y)$ for all x in $\mathcal{U}$ and y in $\mathcal{V}$. An even more general example is obtained by selecting a finite number of elements in $\mathcal{U}'$, say $u_1, \cdots, u_k$, selecting the same finite number of elements in $\mathcal{V}'$, say $v_1, \cdots, v_k$, and writing $w(x, y) = u_1(x)v_1(y) + \cdots + u_k(x)v_k(y)$. Which of the words, "functional" or "form," is used depends somewhat on the context and, somewhat more, on the user's whim. In this book we shall generally use "functional" with "linear" and "form" with "bilinear" (and its higher-dimensional generalizations).

If w_1 and w_2 are bilinear forms on $\mathcal{W}$, and if α_1 and α_2 are scalars, we write w for the function on $\mathcal{W}$ defined by

$$w(x, y) = \alpha_1 w_1(x, y) + \alpha_2 w_2(x, y).$$

It is easy to see that w is a bilinear form; we denote it by $\alpha_1 w_1 + \alpha_2 w_2$. With this definition of the linear operations, the set of all bilinear forms on $\mathcal{W}$ is a vector space. The chief purpose of the remainder of this section is to determine (in the finite-dimensional case) how the dimension of this space depends on the dimensions of $\mathcal{U}$ and $\mathcal{V}$.

THEOREM 1. *If $\mathcal{U}$ is an n-dimensional vector space with basis $\{x_1, \cdots, x_n\}$, if $\mathcal{V}$ is an m-dimensional vector space with basis $\{y_1, \cdots, y_m\}$, and if $\{\alpha_{ij}\}$ is any set of nm scalars ($i = 1, \cdots, n$; $j = 1, \cdots, m$), then there is one and only one bilinear form w on $\mathcal{U} \oplus \mathcal{V}$ such that $w(x_i, y_j) = \alpha_{ij}$ for all i and j.*

PROOF. If $x = \sum_i \xi_i x_i$, $y = \sum_j \eta_j y_j$, and w is a bilinear form on $\mathcal{U} \oplus \mathcal{V}$ such that $w(x_i, y_j) = \alpha_{ij}$, then

$$w(x, y) = \sum_i \sum_j \xi_i \eta_j w(x_i, y_j) = \sum_i \sum_j \xi_i \eta_j \alpha_{ij}.$$

From this equation the uniqueness of w is clear; the existence of a suitable w is proved by reading the same equation from right to left, that is, defining w by it. (Compare this result with § 15, Theorem 1.)

THEOREM 2. *If $\mathcal{U}$ is an n-dimensional vector space with basis $\{x_1, \cdots, x_n\}$, and if $\mathcal{V}$ is an m-dimensional vector space with basis $\{y_1, \cdots, y_m\}$, then there is a uniquely determined basis $\{w_{pq}\}$ $(p = 1, \cdots, n; q = 1, \cdots, m)$ in the vector space of all bilinear forms on $\mathcal{U} \oplus \mathcal{V}$ with the property that $w_{pq}(x_i, x_j) = \delta_{ip}\delta_{jq}$. Consequently the dimension of the space of bilinear forms on $\mathcal{U} \oplus \mathcal{V}$ is the product of the dimensions of $\mathcal{U}$ and $\mathcal{V}$.*

PROOF. Using Theorem 1, we determine w_{pq} (for each fixed p and q) by the given condition $w_{pq}(x_i, y_j) = \delta_{ip}\delta_{jq}$. The bilinear forms so determined are linearly independent, since

$$\sum_p \sum_q \alpha_{pq} w_{pq} = 0$$

implies that

$$0 = \sum_p \sum_q \alpha_{pq}\delta_{ip}\delta_{jq} = \alpha_{ij}.$$

If, moreover, w is an arbitrary element of $\mathcal{W}$, and if $w(x_i, y_j) = \alpha_{ij}$, then $w = \sum_p \sum_q \alpha_{pq} w_{pq}$. Indeed, if $x = \sum_i \xi_i x_i$ and $y = \sum_j \eta_j y_j$, then

$$w_{pq}(x, y) = \sum_i \sum_j \xi_i \eta_j \delta_{ip}\delta_{jq} = \xi_p \eta_q,$$

and, consequently,

$$w(x, y) = \sum_i \sum_j \xi_i \eta_j \alpha_{ij} = \sum_p \sum_q \alpha_{pq} w_{pq}(x, y).$$

It follows that the w_{pq} form a basis in the space of bilinear forms; this completes the proof of the theorem. (Compare this result with § 15, Theorem 2.)

EXERCISES

1. (a) If w is a bilinear form on $\mathcal{R}^n \oplus \mathcal{R}^n$, then there exist scalars α_{ij}, $i, j = 1, \cdots, n$, such that if $x = (\xi_1, \cdots, \xi_n)$ and $y = (\eta_1, \cdots, \eta_n)$, then $w(x, y) = \sum_i \sum_j \alpha_{ij}\xi_i\eta_j$. The scalars α_{ij} are uniquely determined by w.
(b) If z is a linear functional on the space of all bilinear forms on $\mathcal{R}^n \oplus \mathcal{R}^n$, then there exist scalars β_{ij} such that (in the notation of (a)) $z(w) = \sum_i \sum_j \alpha_{ij}\beta_{ij}$ for every w. The scalars β_{ij} are uniquely determined by z.

2. A bilinear form w on $\mathcal{U} \oplus \mathcal{V}$ is *degenerate* if, as a function of one of its two arguments, it vanishes identically for some non-zero value of its other argument; otherwise it is *non-degenerate*.
(a) Give an example of a degenerate bilinear form (not identically zero) on $\mathcal{C}^2 \oplus \mathcal{C}^2$.
(b) Give an example of a non-degenerate bilinear form on $\mathcal{C}^2 \oplus \mathcal{C}^2$.

3. If w is a bilinear form on $\mathcal{U} \oplus \mathcal{V}$, if y_0 is in $\mathcal{V}$, and if a function y is defined on $\mathcal{U}$ by $y(x) = w(x, y_0)$, then y is a linear functional on $\mathcal{U}$. Is it true that if w is non-degenerate, then every linear functional on $\mathcal{U}$ can be obtained this way (by a suitable choice of y_0)?

4. Suppose that for each x and y in $\mathcal{P}_n$ the function w is defined by

(a) $w(x, y) = \int_0^1 x(t)y(t)\,dt,$

(b) $w(x, y) = x(1) + y(1),$

(c) $w(x, y) = x(1) \cdot y(1),$

(d) $w(x, y) = x(1)\left(\frac{dy}{dt}\right)_{t=1}$

In which of these cases is w a bilinear form on $\mathcal{P}_n \oplus \mathcal{P}_n$? In which cases is it non-degenerate?

5. Does there exist a vector space $\mathcal{V}$ and a bilinear form w on $\mathcal{V} \oplus \mathcal{V}$ such that w is not identically zero but $w(x, x) = 0$ for every x in $\mathcal{V}$?

6. (a) A bilinear form w on $\mathcal{V} \oplus \mathcal{V}$ is *symmetric* if $w(x, y) = w(y, x)$ for all x and y. A *quadratic form* on $\mathcal{V}$ is a function q on $\mathcal{V}$ obtained from a bilinear form w by writing $q(x) = w(x, x)$. Prove that if the characteristic of the underlying scalar field is different from 2, then every symmetric bilinear form is uniquely determined by the corresponding quadratic form. What happens if the characteristic is 2?

(b) Can a non-symmetric bilinear form define the same quadratic form as a symmetric one?

§ 24. Tensor products

In this section we shall describe a new method of putting two vector spaces together to make a third, namely, the formation of their tensor product. Although we shall have relatively little occasion to make use of tensor products in this book, their theory is closely allied to some of the subjects we shall treat, and it is useful in other related parts of mathematics, such as the theory of group representations and the tensor calculus. The notion is essentially more complicated than that of direct sum; we shall therefore begin by giving some examples of what a tensor product should be, and the study of these examples will guide us in laying down the definition.

Let $\mathcal{U}$ be the set of all polynomials in one variable s, with, say, complex coefficients; let $\mathcal{V}$ be the set of all polynomials in another variable t; and, finally, let $\mathcal{W}$ be the set of all polynomials in the two variables s and t. With respect to the obvious definitions of the linear operations, $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are all complex vector spaces; in this case we should like to call $\mathcal{W}$, or something like it, the tensor product of $\mathcal{U}$ and $\mathcal{V}$. One reason for this terminology is that if we take any x in $\mathcal{U}$ and any y in $\mathcal{V}$, we may form their product, that is, the element z of $\mathcal{W}$ defined by $z(s, t) = x(s)y(t)$.

(This is the ordinary product of two polynomials. Here, as before, we are doggedly ignoring the irrelevant fact that we may even multiply together two elements of $\mathcal{U}$, that is, that the product of two polynomials in the same variable is another polynomial in that variable. Vector spaces in which a decent concept of multiplication is defined are called *algebras*, and their study, as such, lies outside the scope of this book.)

In the preceding example we considered vector spaces whose elements are functions. We may, if we wish, consider the simple vector space $\mathcal{C}^n$ as a collection of functions also; the domain of definition of the functions is, in this case, a set consisting of exactly n points, say the first n (strictly) positive integers. In other words, a vector $(\xi_1, \cdots, \xi_n)$ may be considered as a function ξ whose value $\xi(i)$ is defined for $i = 1, \cdots, n$; the definition of the vector operations in $\mathcal{C}^n$ is such that they correspond, in the new notation, to the ordinary operations performed on the functions ξ. If, simultaneously, we consider $\mathcal{C}^m$ as the collection of functions η whose value $\eta(j)$ is defined for $j = 1, \cdots, m$, then we should like the tensor product of $\mathcal{C}^n$ and $\mathcal{C}^m$ to be the set of all functions ζ whose value $\zeta(i, j)$ is defined for $i = 1, \cdots, n$ and $j = 1, \cdots, m$. The tensor product, in other words, is the collection of all functions defined on a set consisting of exactly nm objects, and therefore naturally isomorphic to $\mathcal{C}^{nm}$. This example brings out a property of tensor products—namely, the multiplicativity of dimension—that we should like to retain in the general case.

Let us now try to abstract the most important properties of these examples. The definition of direct sum was one possible rigorization of the crude intuitive idea of writing down, formally, the sum of two vectors belonging to different vector spaces. Similarly, our examples suggest that the tensor product $\mathcal{U} \otimes \mathcal{V}$ of two vector spaces $\mathcal{U}$ and $\mathcal{V}$ should be such that to every x in $\mathcal{U}$ and y in $\mathcal{V}$ there corresponds a "product" $z = x \otimes y$ in $\mathcal{U} \otimes \mathcal{V}$, in such a way that the correspondence between x and z, for each fixed y, as well as the correspondence between y and z, for each fixed x, is linear. (This means, of course, that $(\alpha_1 x_1 + \alpha_2 x_2) \otimes y$ should be equal to $\alpha_1(x_1 \otimes y) + \alpha_2(x_2 \otimes y)$, and that a similar equation should hold for $x \otimes (\alpha_1 y_1 + \alpha_2 y_2)$.) To put it more simply, $x \otimes y$ should define a bilinear (vector-valued) function of x and y.

The notion of formal multiplication suggests also that if u and v are linear functionals on $\mathcal{U}$ and $\mathcal{V}$ respectively, then it is their product w, defined by $w(x, y) = u(x)v(y)$, that should be in some sense the general element of the dual space $(\mathcal{U} \otimes \mathcal{V})'$. Observe that this product is a bilinear (scalar-valued) function of x and y.

§ 25. Product bases

After one more word of preliminary explanation we shall be ready to discuss the formal definition of tensor products. It turns out to be technically preferable to get at $\mathfrak{U} \otimes \mathfrak{V}$ indirectly, by defining it as the dual of another space; we shall make tacit use of reflexivity to obtain $\mathfrak{U} \otimes \mathfrak{V}$ itself. Since we have proved reflexivity for finite-dimensional spaces only, we shall restrict the definition to such spaces.

DEFINITION. The *tensor product* $\mathfrak{U} \otimes \mathfrak{V}$ of two finite-dimensional vector spaces $\mathfrak{U}$ and $\mathfrak{V}$ (over the same field) is the dual of the vector space of all bilinear forms on $\mathfrak{U} \oplus \mathfrak{V}$. For each pair of vectors x and y, with x in $\mathfrak{U}$ and y in $\mathfrak{V}$, the tensor product $z = x \otimes y$ of x and y is the element of $\mathfrak{U} \otimes \mathfrak{V}$ defined by $z(w) = w(x, y)$ for every bilinear form w.

This definition is one of the quickest rigorous approaches to the theory, but it does lead to some unpleasant technical complications later. Whatever its disadvantages, however, we observe that it obviously has the two desired properties: it is clear, namely, that dimension is multiplicative (see § 23, Theorem 2, and § 15, Theorem 2), and it is clear that $x \otimes y$ depends linearly on each of its factors.

Another possible (and deservedly popular) definition of tensor product is by formal products. According to that definition $\mathfrak{U} \otimes \mathfrak{V}$ is obtained by considering all symbols of the form $\sum_i \alpha_i(x_i \otimes y_i)$, and, within the set of such symbols, making the identifications demanded by the linearity of the vector operations and the bilinearity of tensor multiplication. (For the purist: in this definition $x \otimes y$ stands merely for the ordered pair of x and y; the multiplication sign is just a reminder of what to expect.) Neither definition is simple; we adopted the one we gave because it seemed more in keeping with the spirit of the rest of the book. The main disadvantage of our definition is that it does not readily extend to the most useful generalizations of finite-dimensional vector spaces, that is, to modules and to infinite-dimensional spaces.

For the present we prove only one theorem about tensor products. The theorem is a further justification of the product terminology, and, incidentally, it is a sharpening of the assertion that dimension is multiplicative.

THEOREM. *If $\mathfrak{X} = \{x_1, \cdots, x_n\}$ and $\mathfrak{Y} = \{y_1, \cdots, y_m\}$ are bases in $\mathfrak{U}$ and $\mathfrak{V}$ respectively, then the set $\mathfrak{Z}$ of vectors $z_{ij} = x_i \otimes y_j$ $(i = 1, \cdots, n;$ $j = 1, \cdots, m)$ is a basis in $\mathfrak{U} \otimes \mathfrak{V}$.*

PROOF. Let w_{pq} be the bilinear form on $\mathfrak{U} \oplus \mathfrak{V}$ such that $w_{pq}(x_i, y_j) = \delta_{ip}\delta_{jq}$ $(i, p = 1, \cdots, n; j, q = 1, \cdots, m)$; the existence of such bilinear forms, and the fact that they constitute a basis for all bilinear forms, follow from § 23, Theorem 2. Let $\{w'_{pq}\}$ be the dual basis in $\mathcal{U} \otimes \mathcal{V}$, so that $[w_{ij}, w'_{pq}] = \delta_{ip}\delta_{jq}$. If $w = \sum_p \sum_q \alpha_{pq} w_{pq}$ is an arbitrary bilinear form on $\mathcal{U} \oplus \mathcal{V}$, then

$$w'_{ij}(w) = [w, w'_{ij}] = \sum_p \sum_q \alpha_{pq}[w_{pq}, w'_{ij}]$$
$$= \alpha_{ij} = w(x_i, y_j) = z_{ij}(w).$$

The conclusion follows from the fact that the vectors w'_{ij} do constitute a basis of $\mathcal{U} \otimes \mathcal{V}$.

EXERCISES

1. If $x = (1, 1)$ and $y = (1, 1, 1)$ are vectors in $\mathcal{R}^2$ and $\mathcal{R}^3$ respectively, find the coordinates of $x \otimes y$ in $\mathcal{R}^2 \otimes \mathcal{R}^3$ with respect to the product basis $\{x_i \otimes y_j\}$, where $x_i = (\delta_{i1}, \delta_{i2})$ and $y_j = (\delta_{1j}, \delta_{2j}, \delta_{3j})$.

2. Let $\mathcal{P}_{n,m}$ be the space of all polynomials z with complex coefficients, in two variables s and t, such that either $z = 0$ or else the degree of $z(s, t)$ is $\leqq m - 1$ for each fixed s and $\leqq n - 1$ for each fixed t. Prove that there exists an isomorphism between $\mathcal{P}_n \otimes \mathcal{P}_m$ and $\mathcal{P}_{n,m}$ such that the element z of $\mathcal{P}_{n,m}$ that corresponds to $x \otimes y$ (x in $\mathcal{P}_n$, y in $\mathcal{P}_m$) is given by $z(s, t) = x(s)y(t)$.

3. To what extent is the formation of tensor products commutative and associative? What about the distributive law $\mathcal{U} \otimes (\mathcal{V} \oplus \mathcal{W}) = (\mathcal{U} \otimes \mathcal{V}) \oplus (\mathcal{U} \otimes \mathcal{W})$?

4. If $\mathcal{V}$ is a finite-dimensional vector space, and if x and y are in $\mathcal{V}$, is it true that $x \otimes y = y \otimes x$?

5. (a) Suppose that $\mathcal{V}$ is a finite-dimensional real vector space, and let $\mathcal{U}$ be the set $\mathcal{C}$ of all complex numbers regarded as a (two-dimensional) real vector space. Form the tensor product $\mathcal{V}^+ = \mathcal{U} \otimes \mathcal{V}$. Prove that there is a way of defining products of complex numbers with elements of $\mathcal{V}^+$ so that $\alpha(x \otimes y) = \alpha x \otimes y$ whenever α and x are in $\mathcal{C}$ and y is in $\mathcal{V}$.

(b) Prove that with respect to vector addition, and with respect to complex scalar multiplication as defined in (a), the space $\mathcal{V}^+$ is a complex vector space.

(c) Find the dimension of the complex vector space $\mathcal{V}^+$ in terms of the dimension of the real vector space $\mathcal{V}$.

(d) Prove that the vector space $\mathcal{V}$ is isomorphic to a subspace in $\mathcal{V}^+$ (when the latter is regarded as a real vector space).

The moral of this exercise is that not only can every complex vector space be regarded as a real vector space, but, in a certain sense, the converse is true. The vector space $\mathcal{V}^+$ is called the *complexification* of $\mathcal{V}$.

6. If $\mathcal{U}$ and $\mathcal{V}$ are finite-dimensional vector spaces, what is the dual space of $\mathcal{U}' \otimes \mathcal{V}'$?

§ 26. Permutations

The main subject of this book is usually known as linear algebra. In the last three sections, however, the emphasis was on something called multilinear algebra. It is hard to say exactly where the dividing line is between