Section 16.

Turn other type of eqn into Bessel eqn.

$$x(xy')' + (x^2 - p^2)y = 0.$$

(16.1) $$y'' + \frac{1-2a}{x} y' + \left[(bc \cdot x^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2} \right] \cdot y = 0.$$

has solution

(16.2) $$y = x^a \cdot Z_p(b \cdot x^c)$$

$$Z_p = J_p, \quad N_p.$$

$$N_p(x) = \frac{\cos(p \cdot \pi) \cdot J_p(x) - J_{-p}(x)}{\sin(p\pi)}.$$

(for $p \notin \mathbb{Z}$, otherwise take limit).

For example: Exercise 16.2.

$$y'' + 4x^2 y = 0$$

Try to identify parameters:

$$y'' + \frac{1-2a}{x} y' + \left[(bc \cdot x^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2} \right] \cdot y = 0.$$

$$\begin{cases} 1 - 2a = 0 \\ (bc \cdot x^{c-1})^2 = 4x^2 \\ a^2 - p^2c^2 = 0 \end{cases}$$

$a = \frac{1}{2}$

$P = \frac{1}{4}$

$(\frac{1}{2})^2 - p^2 \cdot 2^2 = 0$

$2(c-1) = 2 \Rightarrow c = 2, \quad b = 1$

$(bc \cdot x^{c-1})^2 = (1 \cdot 2 \cdot x^1)^2 = 4x^2$ ✓

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so the solution is.

$$y = x^{\frac{1}{2}} Z_{\frac{1}{4}}(1 \cdot x^2).$$

it has 2 gen sol'n.

$$x^{\frac{1}{2}} \cdot J_{\frac{1}{4}}(x^2), \quad x^{\frac{1}{2}} N_{\frac{1}{4}}(x^2)$$

Let's verify that the sol'n satisfies the eqn:

$$y = x^a \cdot Z_p(bx^c)$$

$$\frac{y}{x^a} = Z_p(bx^c).$$

$$\left(bx^c \frac{d}{d(bx^c)}\right)\left(bx^c \frac{d}{d(bx^c)}\right)\left(\frac{y}{x^a}\right) + \left((bx^c)^2 - p^2\right)\frac{y}{x^a} = 0$$

$$bx^c \frac{d}{d(bx^c)} = bx^c \frac{d}{cbx^{c-1} \cdot dx} = \left(\frac{1}{c}\right) x \frac{d}{dx}$$

$$d(bx^c) = b \cdot x^{c-1} \cdot c \cdot dx.$$

$$\frac{1}{c^2}\left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}\right)\left(\frac{y}{x^a}\right)$$

$$= \frac{1}{c^2} \frac{1}{x^a}\left(x\frac{d}{dx} - a\right)\left(x\frac{d}{dx} - a\right) y.$$

$$\left(x\frac{d}{dx}\right)\left(x^\alpha \cdot y\right)$$

$$= x\frac{d}{dx}(x^\alpha) \cdot y + x^\alpha \cdot x\frac{d}{dx}(y)$$

$$= \alpha \cdot x^\alpha \cdot y + x^\alpha x \frac{d}{dx} y$$

$$= x^\alpha \left(x\frac{d}{dx} + \alpha\right) y$$

$Z_p(u) \qquad u = bx^c.$

$$u\frac{d}{du} \cdot u\frac{d}{du} \cdot Z_p(u) + (u^2 - p^2)\, Z_p(u) = 0$$

replace u by bx^c

$Z_p(u)$ by $\frac{y}{x^a}.$

$$\frac{1}{c^2}\, x^{-a} \cdot \left(x\frac{d}{dx} - a\right)^2 \cdot (y) + ((bx^c)^2 - p^2)\, \frac{y}{x^a} = 0$$

$$\left(x\frac{d}{dx} - a\right)^2 y + \left[(bcx^c)^2 - p^2c^2\right] y = 0$$

$$\left[\left(x\frac{d}{dx}\right)^2 y - 2a \cdot x\frac{d}{dx} \cdot y + a^2 y\right] + \left[(bcx^c)^2 - p^2c^2\right] y = 0$$

....

Exercises: 3, 5, 7, 9.

Section 19 Orthogonality, & Wronskian.

$$x(xy')' + (x^2 - p^2)\, y = 0$$

divide by x,

$$\frac{d}{dx}\left(x\frac{d}{dx}y\right) + (x^2 - p^2)\frac{y}{x} = 0.$$

suppose y_1 and y_2 satisfies the same equation.

$$\begin{cases} \frac{d}{dx}\left(x\frac{d}{dx}y_1\right) + (x^2 - p^2)\frac{y_1}{x} = 0 & (1) \\ \frac{d}{dx}\left(x\frac{d}{dx}y_2\right) + (x^2 - p^2)\frac{y_2}{x} = 0 & (2) \end{cases}$$

$(1) \cdot y_2 - (2) \cdot y_1.$

$$\underbrace{y_2 \cdot \frac{d}{dx}\left(x\frac{d}{dx}y_1\right)}_{\mathrm{I}} - \underbrace{y_1\frac{d}{dx}\left(x\frac{d}{dx} \cdot y_2\right)}_{\mathrm{II}} = 0.$$

$$f \cdot \frac{d}{dx}(g) = \frac{d}{dx}(f \cdot g) - \frac{d}{dx}(f) \cdot g.$$

$$I = \frac{d}{dx}\left\{ \underbrace{y_2}\cdot \underbrace{x \cdot \frac{d}{dx} y_1} \right\} - \frac{d}{dx}(y_2)\cdot x \frac{d}{dx}\cdot y_1.$$

$$= \frac{d}{dx}\{x \cdot y_1' \cdot y_2\} - \underline{x \cdot y_1' \cdot y_2'}$$

$$II = \frac{d}{dx}\{x \cdot y_2' \cdot y_1\} - \underline{x \cdot y_2' \cdot y_1'}$$

$$I - II = \frac{d}{dx}\{x \cdot y_1' \cdot y_2 - x\, y_2' \cdot y_1\} = 0.$$

$$\Rightarrow \quad x \cdot y_1' \cdot y_2 - x \cdot y_2' \cdot y_1 = C$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' = \frac{-C}{x}$$

<u>Claim</u>

$$W(\underline{J_p(x)}, \underline{J_{-p}(x)}) = -\frac{2}{\pi}\cdot \sin(p\pi) / x$$

indeed, if p is integer. $W(J_p, J_{-p}) = 0$

- We are considering functions of type

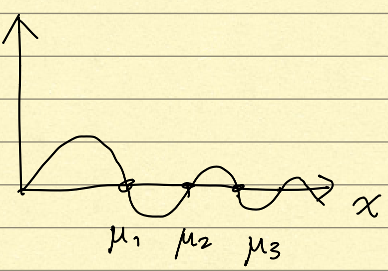
$y(x) = \underline{J_p(\alpha x)}$ for fixed p ~~by~~ but varying α.

it satisfies equation.

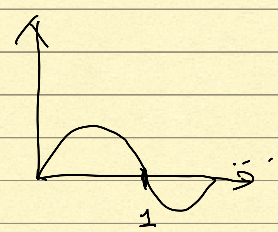
$$x(x y')' + (\alpha^2 x^2 - p^2)\cdot y = 0.$$

we will consider only those $\underline{\alpha}$, s.t. $\underline{y(1) = 0}$.

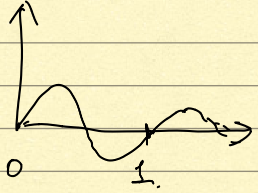
$J_p(x)$



$J_p(\frac{x}{\mu_1})$



$J_p\left(\frac{x}{\mu_2}\right)$:



$\sin(nx)$

Suppose α, β are chosen s.t. $J_p(\alpha) = 0$, $J_p(\beta) = 0$.

then we want to prove that

$$\int_0^1 x \cdot J_p(\alpha x) \cdot J_p(\beta x)\, dx = 0.$$

$J_p(\alpha x)$ satisfies. $\quad x \frac{d}{dx}\left(x \cdot \frac{d}{dx}\left(J_p(\alpha x)\right)\right) + (\alpha^2 x^2 - p^2) J_p(\alpha x) = 0 \quad (1)$

$J_p(\beta x) \quad x \frac{d}{dx}\left(x \frac{d}{dx}\left(J_p(\beta x)\right)\right) + (\beta^2 x^2 - p^2) J_p(\beta x) = 0 \quad (2)$

$$\frac{(1)}{x} \cdot J_p(\beta x) - \frac{(2)}{x} \cdot J_p(\alpha x) = 0$$

LHS. $\quad \underbrace{J_p(\beta x) \frac{d}{dx}\left(x \cdot \frac{d}{dx}\left(J_p(\alpha x)\right)\right)}_{\mathrm{I}} - \underbrace{J_p(\alpha x) \frac{d}{dx}\left(x \frac{d}{dx} J_p(\beta x)\right)}_{\mathrm{II}}$

$+ \underline{(\alpha^2 - \beta^2) \cdot x \cdot J_p(\alpha x) J_p(\beta x)} = 0.$

$$\int_0^1 \text{LHS}\, dx = 0$$

$$\int_0^1 \mathrm{I} - \mathrm{II}\, dx = \int_0^1 \frac{d}{dx}\left(x \cdot J_p(\alpha x) \frac{d}{dx} J_p(\beta x) - x J_p(\beta x) \frac{d}{dx} J_p(\alpha x)\right) dx = 0$$

$\neq 0$

$$(\alpha^2 - \beta^2) \underbrace{\int_0^1 x \cdot J_p(\alpha x) J_p(\beta x)\, dx = 0}_{=0.}$$