Ch 11 : only Γ function, β- function.

- definitions. relate β to Γ function.
  use them to do definite integrals.

(1) $\Gamma(p) = \int_0^\infty x^{p-1} \cdot e^{-x} \cdot dx$ $\quad \underline{p>0}$

$\Gamma(p+1) = \Gamma(p) \cdot p. \iff \Gamma(p) = \frac{1}{p}\Gamma(p+1).$

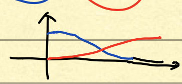
if $p = n+1$, $n \in \mathbb{N}$.

$\Gamma(n+1) = n!$

(2). $B(p,q) = \int_0^1 (x)^{p-1} (1-x)^{q-1}\, dx$ $\quad p>0,\ q>0.$

$= 2\int_0^{\pi/2} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1} \cdot d\theta.$

$= \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\, dy.$ $\quad \frac{1}{1+y},\ \frac{y}{1+y}$



$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

$\left(\text{c.f. } \binom{n}{m} = \frac{n!}{m!(n-m)!} \right)$

Ex:

11.5.6. Prove

$$\frac{d}{dp}\Gamma(p) = \int_0^\infty x^{p-1} \cdot e^{-x} \cdot \ln x \cdot dx.$$

Pf:

$$\Gamma(p) = \int_0^\infty x^{p-1} \cdot e^{-x} \cdot dx.$$

$$\frac{d}{dp}\Gamma(p) = \frac{d}{dp}\int_0^\infty x^{p-1} \cdot e^{-x} \cdot dx$$

$$= \int_0^\infty \underline{\frac{d}{dp}(x^{p-1})} \cdot e^{-x} \cdot dx$$

$$= \int_0^\infty x^{p-1} \cdot \underline{\ln x} \cdot e^{-x} \cdot dx.$$

$$\frac{d}{dp}(\alpha^p) = \frac{d}{dp}\left(e^{p \cdot \ln\alpha}\right)$$

$$= (\ln\alpha) \cdot e^{p \cdot \ln\alpha}$$

$$= (\ln\alpha) \cdot \alpha^p$$

11.3.12

$$\int_0^\infty x \cdot e^{-x^3} dx.$$

do a change of variable: $\quad x>0,\ u>0$

$$u = x^3, \Rightarrow x = u^{\frac{1}{3}}$$

$$du = 3 \cdot x^2 \cdot dx$$

$$\int_0^\infty x \cdot e^{-u} \cdot \left(\frac{du}{3x^2}\right)$$

$$= \int_0^\infty \frac{1}{3} \frac{1}{x} \cdot e^{-u}\, du$$

$$= \frac{1}{3}\int_0^\infty u^{-\frac{1}{3}} \cdot e^{-u}\, du$$

$$= \frac{1}{3}\underline{\int_0^\infty u^{\frac{2}{3}-1} e^{-u}\, du} = \frac{1}{3}\Gamma\left(\frac{2}{3}\right)$$

11.3.17.

$$\int_0^1 \left[\underline{\ln\left(\frac{1}{x}\right)}\right]^{p-1} dx.$$

x $0 \to 1$. $\quad \frac{1}{x} : \infty \to 1$ $\quad \ln\left(\frac{1}{x}\right) : \infty \to 0$

$$u = \ln\left(\frac{1}{x}\right), \quad e^u = \frac{1}{x}, \quad e^{-u} = x.$$

$$I = \int_\infty^0 u^{p-1} \frac{d(e^{-u})}{du} \cdot du$$

$$= (-1)\int_0^\infty u^{p-1}(-e^{-u})\, du$$

$$= \int_0^\infty u^{p-1} e^{-u}\, du = \Gamma(p).$$

11.7.5.

$$\int_0^\infty \frac{y^2}{(1+y)^6} \cdot dy.$$

$p-1=2,\ p+q=6 \Rightarrow p=3,\ q=3$

$$= B(3,3) = \frac{\Gamma(3)\Gamma(3)}{\Gamma(6)} = \frac{2!\,2!}{5!}$$

- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)}.$$ valid whenever $p \notin \mathbb{Z}$.

Ch 12:

- Equation.
- generating function.
- integral expression (like contour integral).
- orthogonality conditions.
- recursion relation.
- Series expansion.

only test on Legendres + Bessel.

(section 22 not in exam).

Legendre function: (Associate Legendre fun

- $P_l(x)$. $P_l^m(x)$. $(m \leq l)$.

$$(1-x^2)\, y'' - 2x \cdot y' + \left[l(l+1) - \frac{m^2}{1-x^2} \right] y = 0$$

(take $m=0$. you get eq for Legendre function)

if $u(\theta) = P_l^m(\cos\theta)$, then u. satisfies. (eigenvalue problem for Δ on S^2).

(Problem 12.10.2)

$$\frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{du}{d\theta} \right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta} \right] \cdot u = 0.$$

- Orthogonality condition:

$U_{l,m}(\theta,\phi) = P_l^m(\cos\theta) \cdot \cos(m\phi)$

for different (l,m). $U_{l,m}(\theta,\phi)$ are orthogonal w.r.t. area form on S^2.

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} U_{l,m}(\theta,\phi) \cdot U_{l',m'}(\theta,\phi) \sin\theta \, d\theta \, d\phi = 0 \quad \text{if } l \neq l' \text{ or } m \neq m'.$$

In particular, if $m = m'$, $l \neq l'$.

$$\int_{\theta=0}^{\pi} \sin\theta \cdot P_l^m(\cos\theta) \cdot P_{l'}^m(\cos\theta) \, d\theta = 0.$$

(12.10.3) Orthogonality for P_l^m

Eq. (10.3). $u = P_l^m(x)$, $v = P_n^m(x)$.

① $(1-x^2) \cdot u'' - 2x \cdot u' + \left[l(l+1) - \frac{m^2}{1-x^2} \right] u = 0$

② $(1-x^2) \cdot v'' - 2x \cdot v' + \left[n(n+1) - \frac{m^2}{1-x^2} \right] \cdot v = 0$

①$\cdot v$ − ②$\cdot u$.

$(1-x^2)\,[u'' \cdot v - v'' \cdot u] - 2 \cdot x \cdot (u'v - v'u)$
$+ [l(l+1) - n(n+1)] \cdot uv = 0.$

Goal: $\int_{-1}^{1} (1-x^2) \cdot (u''v - v''u) - 2 \cdot x \cdot (u'v - v'u)\, dx = 0.$

(Refer Section 7.)

Rodrigue Formula:

$$P_l(x) = \frac{1}{2^l \cdot l!} \frac{d^l}{dx^l} \cdot (x^2-1)^l.$$

$$P_l^m(x) = (1-x^2)^{\frac{m}{2}} \cdot \frac{d^m}{dx^m} \cdot P_l(x)$$

Generating Fcn:

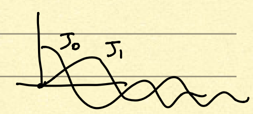
$|h| < 1$.

$$\Phi(x,h) = \sum_{n=0}^{\infty} h^n \cdot P_n(x) = \frac{1}{\sqrt{1 - 2x \cdot h + h^2}}$$

Bessel Function Recursion Relation (Sec 15).

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+p)!} \left(\frac{x}{2} \right)^{2n+p}. \quad (p \text{ integer}, p \geq 0)$$

$$= \frac{1}{p!} \left(\frac{x}{2} \right)^p + \cdots$$



$$\begin{cases} x^{-p} \cdot \frac{d}{dx}\left(x^p \cdot J_p(x) \right) = J_{p-1}(x) & ✓ \\ x^{+p} \frac{d}{dx}\left(x^{-p} \cdot J_p(x) \right) = -J_{p+1}(x) & ✓ \end{cases}$$

$$x^{-p} \frac{d}{dx}\left(x^p \cdot J_p \right) = \left(\frac{d}{dx} + \frac{p}{x} \right) \cdot J_p.$$

$$x^{+p} \frac{d}{dx}\left(x^{-p} J_p \right) = \left(\frac{d}{dx} - \frac{p}{x} \right) J_p.$$

(follow the book.)

Ex 7.

$$\int_0^\infty J_1(x)\, dx = -J_0(x) \Big|_0^\infty = 1.$$

$p=0$, $x^{+p} \cdot \frac{d}{dx}(x^{-p} \cdot J_p) = -J_{p+1}$

$\frac{d}{dx} \cdot J_0 = -J_1(x).$

$J_0(x=0) = 1$

$J_p(\infty) = 0$

$J_p(x) \sim \frac{1}{\sqrt{x}} \cdot \cos(\cdot x \cdot)$ $_{x \to \infty}$

$$\int_0^\infty J_3(x) - J_1(x)\, dx \sim J_2(x)\Big|_0^\infty = 0 - 0 = 0$$

$$J_2(x) \sim \frac{1}{2!}\left(\frac{x}{2}\right)^2 \to 0 \quad x \to 0$$

As $x \to \infty$. $J_p(x) \approx \sqrt{\frac{2}{\pi x}} \cdot \cos\left(x - \frac{2p+1}{4}\pi\right) \to 0.$

$$\int_0^\infty J_2(x) - J_0(x)\, dx \sim J_1(x)\Big|_0^\infty = 0 - 0 = 0$$

$$\int_0^\infty J_0(x)\, dx = \int_0^\infty J_0(x) \cdot e^{-px}\, dx\Big|_{p=0}$$

$$\overset{L.T.}{=} \frac{1}{\sqrt{1+p^2}}\Big|_{p=0} = 1.$$