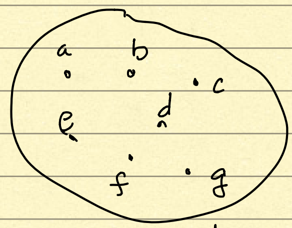
Sample space : a set Ω.

outcomes : element in Ω

event : subset in Ω.

Ex: $\Omega =$



- a set of 7 elements.
- d is an outcome.
- $A = \{a, c, e\} \subset \Omega$ is an event.

or $\phi \subset \Omega$. 2^7 possible subsets in Ω.

$\Omega \subset \Omega$.

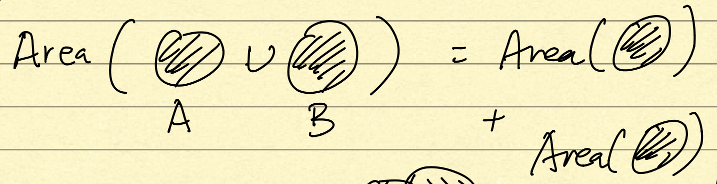
Probability: is a way assign a real number to each event.

If A is an event, we use $\mathbb{P}(A)$ to denote the number assigned. "Probability for event A to happen".

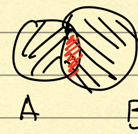
Axioms for $\mathbb{P}$:

- $0 \le \mathbb{P}(A) \le 1$
- $\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\phi) = 0$ (empty set)
- If $A, B \subset \Omega$, $A \cap B = \phi$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

if we compare $\mathbb{P}$ as area.



- Consequence:



$$\mathbb{P}(A \cup B) = \mathbb{P}((A \setminus B) \cup (B \setminus A) \cup (A \cap B))$$
$$= \mathbb{P}(A \setminus B) + \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B)$$
$$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

- **Independence**: Let A, B be events, we say A and B are indep. if

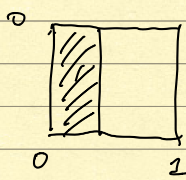
$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Ex: choose a number x uniformly in $[0,1]$, choose a number y uniform in $[0,1]$.

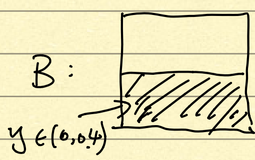
$A := x \in (0, 0.3)$

$B := y \in (0, 0.4)$

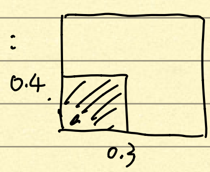
A:



, B:



$A \cap B$:



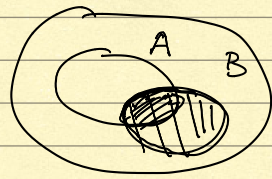
$\therefore$ $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

Hence A is indep of B.

- **Conditional Probability**:

A, B two events:

$\mathbb{P}(A \mid B) :=$ what is the probability that A happens given that B happened.



$$= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

"renormalization"

Ex: 4 balls, $\overset{1}{B}, \overset{2}{B}, \overset{3}{W}, \overset{4}{W}$ in the bag. You pull out 2 balls.

$$\Omega = \{BB, BW, WB, WW\}.$$

pull out 2 balls in order, there are $4 \times 3 = 12$ ways, equally likely.

$BB = \{①②, ②①\}$, $\mathbb{P}(BB) = \frac{2}{12} = \frac{1}{6}$

$BW = \{①③, ①④, ②③, ②④\}$, $\mathbb{P}(BW) = \frac{4}{12} = \frac{1}{3}$.

another way to say A and B are indep.

$\mathbb{P}(A \mid B) = \mathbb{P}(A) \iff A, B$ indep

- **Product Rule:**

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B \mid A) = \mathbb{P}(B) \cdot \mathbb{P}(A \mid B)$$

- **Baysian Formula:**

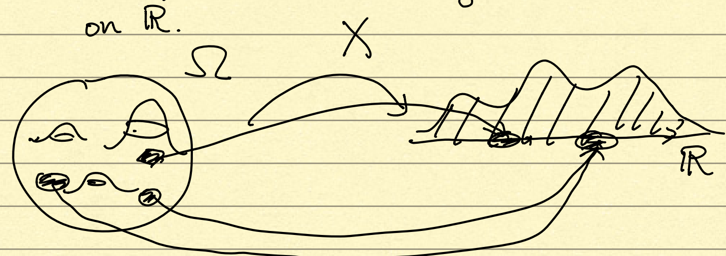
$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(B) \cdot \mathbb{P}(A \mid B)}{\mathbb{P}(A)}.$$

Random Variable:

Def: a function on the sample space:

$$X : \Omega \to \mathbb{R}$$

- "push forward probability measure" on $\mathbb{R}$.



$\mathbb{P}$ on Ω.

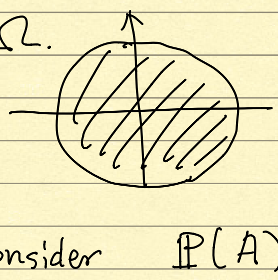
"heap of sand"

$\mathbb{P}_X$

Then, we get a probability measure on $\mathbb{R}$.

Ex: Consider Ω = unit disk in $\mathbb{R}^2$.

Ω.



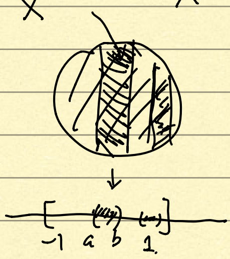
consider $\mathbb{P}(A) := \dfrac{\text{Area}(A)}{\text{Area}(\Omega)}$

consider:

$X : \Omega \to \mathbb{R}$

$(x, y) \mapsto x$.

X^{-1}



then we get a probability on $[-1, 1]$. $\mathbb{P}_X$

WB, $\mathbb{P}(WB) = \frac{1}{3}$, $\mathbb{P}(WW) = \frac{1}{6}$.

$\frac{1}{6} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} = 1$ ✓.

- Two events: $A_1 = \{\text{first ball is black}\}$

$A_2 = \{\text{2nd ball is black}\}$.

$$\frac{1}{3} = \mathbb{P}(A_2 \mid A_1) = \frac{\mathbb{P}(A_1 \cap A_2)}{\mathbb{P}(A_1)} = \frac{\mathbb{P}(\{BB\})}{\mathbb{P}(\{BB, BW\})} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{1+2} = \frac{1}{3}.$$

The $\mathbb{P}_X$ probability on $\mathbb{R}$ is defined as: $I = (a, b) \subset \mathbb{R}$

$$\mathbb{P}_X(I) := \mathbb{P}_\Omega(X^{-1}(I))$$

one can check that $\mathbb{P}_X$ is a probability on the sample space $\mathbb{R}$.

$$X : (\Omega, \mathbb{P}_\Omega) \to (\mathbb{R}, \mathbb{P}_X)$$

Ex: $\Omega = \{HHH, HHT, HTH, \cdots\}$ $2^3 = 8$ of them

an event is any subset of Ω.

$A = \{HHH, TTT, THT\}$ is an event

$$\mathbb{P}_X((a, b)) = \mathbb{P}_\Omega(\{(x, y) \in \Omega \mid x \in (a, b)\}) = \mathbb{P}(X^{-1}((a, b)))$$