Today:

① About Γ function and β-functions. ✓

(optional) ② Contour integral expression for Γ, β, Bessel functions.

③ generating functions for Bessel functions.

① Beta function. (Ch 11.6).

Def: $$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \qquad (\mathrm{I})$$

$p>0,\ q>0.$

($\because \int_0^{\varepsilon} \frac{1}{x}\,dx$ is divergent, we want $p>0, q>0$).

Change $x \leftrightarrow 1-x$, we see

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \int_0^1 (1-x)^{p-1} x^{q-1}\,dx = B(q,p).$$

Another expression: let $x = \sin^2\theta$, then $\theta \in (0, \frac{\pi}{2})$

$$dx = 2\sin\theta\cdot\cos\theta\cdot d\theta.$$
$$1-x = 1-\sin^2\theta = \cos^2\theta.$$

$$B(p,q) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{p-1}(\cos^2\theta)^{q-1}\cdot 2\sin\theta\cdot\cos\theta\cdot d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\cdot d\theta. \qquad (\mathrm{II})$$

A third way to write $B(p,q)$ as integral, is to write $x = \frac{y}{1+y}$, let y runs from 0 to $+\infty$

$$dx = d\left(\frac{y}{1+y}\right) = d\left(1-\frac{1}{1+y}\right) = \frac{1}{(1+y)^2}\,dy$$

$$B(p,q) = \int_0^\infty \left(\frac{y}{1+y}\right)^{p-1}\left(1-\frac{y}{1+y}\right)^{q-1}\cdot\left(\frac{1}{1+y}\right)^2 dy$$
$$= \int_0^\infty \frac{y^{p-1}}{(1+y)^{p-1+q-1+2}}\cdot dy$$
$$= \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\cdot dy. \qquad (\mathrm{III})$$

§11.7 Relate β-function with Γ function.

Goal: $$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

(or $\mathrm{Re}\,x>0$).

Recall that, for $x>0$,

$$\Gamma(x) = \int_0^\infty t^{x-1}\cdot e^{-t}\cdot dt.$$

let $t = y^2$.

$$= \int_0^\infty (y^2)^{x-1}\cdot e^{-y^2}\,dy^2$$
$$= \int_0^\infty y^{2x-2}\cdot e^{-y^2}\cdot 2y\cdot dy.$$
$$= 2\int_0^\infty y^{2x-1} e^{-y^2}\,dy$$

Rename x as p and x as q, and $\Gamma(\frac{1}{2})$?

$$\Gamma(p)\Gamma(q) = \left(2\int_0^\infty y_1^{2p-1} e^{-y_1^2}\,dy_1\right)\cdot\left(2\cdot\int_0^\infty y_2^{2q-1}\cdot e^{-y_2^2}\,dy_2\right).$$

$$= 4\int_0^\infty\int_0^\infty y_1^{2p-1} y_2^{2q-1} e^{-(y_1^2+y_2^2)}\,dy_1\,dy_2.$$

change to polar coordinate.

$$y_1 = r\cos\theta,\quad y_2 = r\cdot\sin\theta.$$
$$dy_1\,dy_2 = r\cdot dr\cdot d\theta.$$

$$= 4\cdot\int_{r=0}^\infty\int_{\theta=0}^{\frac{\pi}{2}} (r\cos\theta)^{2p-1}\cdot(r\cdot\sin\theta)^{2q-1}\cdot e^{-r^2}\cdot r\,dr\,d\theta$$

$$= 4\cdot\int_0^\infty r^{2p+2q-1} e^{-r^2}\,dr \cdot \int_0^{\frac{\pi}{2}} (\cos\theta)^{2p-1}(\sin\theta)^{2q-1}\cdot d\theta.$$

$$= 2\cdot\Gamma(p+q)\cdot\int_0^{\frac{\pi}{2}}(\cos\theta)^{2p-1}(\sin\theta)^{2p-1}\,d\theta.$$

$$= \Gamma(p+q)\cdot B(p,q)$$

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

(1) $\Gamma(n+1) = n!$

$$B(n+1, m+1) = \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)} = \frac{n!\,m!}{(n+m+1)!}$$

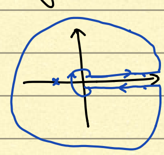
$$\left\{\begin{matrix} n+m \\ n \end{matrix}\right\} = \frac{(n+m)!}{n!\,m!} = \frac{1}{n+m+1}\cdot\frac{(n+m+1)!}{n!\,m!}$$
$$= \frac{1}{n+m+1}\cdot\frac{1}{B(n+1,m+1)}.$$

(2) When $p+q=1$. $\Gamma(1)=1$.

$$B(p,1-p) = \Gamma(p)\Gamma(1-p).$$

$$B(p,1-p) = \int_0^\infty \frac{y^{p-1}}{1+y}\,dy = \cdots = \frac{\pi}{\sin(p\pi)} \qquad (\star)$$

(Boas. Ch 14.7 Ex 5).



$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin(p\pi)} \qquad ☆$$

$$\Rightarrow \quad p = \tfrac{1}{2}. \qquad \Gamma(\tfrac{1}{2})\cdot\Gamma(\tfrac{1}{2}) = \frac{\pi}{\sin(\pi/2)} = \pi$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

(2)
Integral Expression For Bessel Function

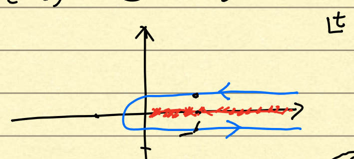
$$J_p(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{2}\right)^{2n+p}}{n!\,\Gamma(n+p+1)}$$

$$\frac{1}{\Gamma(n+p+1)} = \int (?).$$

(now, we will use "Hankel representation for inverse Γ-function")

Consider the integral:

$$I = \int_C (-t)^{z-1}\, e^{-t}\, dt \qquad \mathrm{Re}(z) > 0.$$



(if $z > 0$)

$\because$ $(-t)^{z-1}$ has positive value for t on the negative real axis

above the real axis, we have

if $t = r + i\varepsilon \qquad |\varepsilon| \ll 1.$

$-t = r\cdot e^{-i\pi}$

$$(-t)^{z-1} = (r\cdot e^{-i\pi})^{z-1} = r^{z-1}\cdot e^{-i\pi(z-1)}.$$

if $t = r - i\varepsilon. \quad , \; -t = r\cdot e^{i\pi}.$ $\quad t = r - i\varepsilon$

$$(-t)^{z-1} = (r\cdot e^{i\pi})^{z-1} = r^{z-1}\cdot e^{i\pi(z-1)}.$$

$$\therefore \; I = \int_\infty^0 r^{z-1}\cdot e^{-i\pi(z-1)}\cdot e^{-r}\, dr + \int_0^\infty r^{z-1}\cdot e^{i\pi(z-1)}\cdot e^{-r}\, dr.$$

$$= \left[e^{i\pi(z-1)} - e^{-i\pi(z-1)}\right]\cdot \int_0^\infty e^{-r}\cdot r^{z-1}\, dr.$$

$$= \left(-e^{i\pi z} + e^{-i\pi z}\right)\cdot \Gamma(z).$$

$$= -2i\cdot \sin(\pi z)\cdot \Gamma(z).$$

$$\Gamma(z) = \frac{-1}{2i\cdot\sin(\pi z)}\cdot \int (-t)^{z-1}\cdot e^{-t}\, dt.$$

$$\because \quad \Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

$$\frac{1}{\Gamma(1-z)} = \cdots$$

$$\boxed{\frac{1}{\Gamma(z)} = \frac{i}{2\pi}\cdot \int_\infty^{(0^+)} (-t)^{z}\cdot e^{-t}\, dt}$$

$$J_p(x) = \frac{1}{2\pi i} \int_{-\infty}^{(0^+)} \frac{1}{u^{p+1}}\cdot e^{\frac{1}{2}x\left(u - \frac{1}{u}\right)}\cdot du.$$

in special case $p = n$ interger

$$\boxed{J_n(x) = \frac{1}{2\pi i} \oint_{|u|=1} \frac{1}{u^{p+1}}\, e^{\frac{1}{2}x\left(u-\frac{1}{u}\right)}\cdot du.} \qquad ☆$$