General Strategy to deal with Orthogonal Polynomials

① what equation do they satisfy

② Nice formula to generate them:
like Rodrigue formula.
or Generating functions.

③ Are there any integral expression for them

④ Recursion Relations.

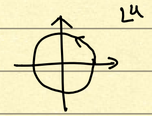
⑤ Ortho Condition. (what inner product ?)

⑥ Asymptotic expansion: $J_n(x) \overset{x\to\infty}{\approx}$?

Recall from last time:

- $$J_n(x) = \oint_{|u|=1} \frac{1}{u^{n+1}} \cdot e^{\frac{1}{2}x(u-\frac{1}{u})} \frac{du}{2\pi i}$$

- Recall also



$$\oint_{|u|=1} \frac{f(u)}{u^{p+1}} \frac{du}{2\pi i} \qquad f(u) : \text{some analytic}$$

$$= \frac{f^{(p)}(0)}{p!}$$

$$\left(\oint \frac{1}{u} \frac{du}{2\pi i} = 1 \right).$$

$$* \quad \cancel{J_n(x) = \frac{1}{n!}\left(\frac{d}{du}\right)^n\Big|_{u=0} \left\{ e^{\frac{1}{2}x(u-\frac{1}{u})} \right\}}$$

wrong! cannot plug in $u=0$.

$$\Leftrightarrow \quad e^{\frac{1}{2}x(u-\frac{1}{u})} = \sum_{n=-\infty}^{\infty} u^n \cdot J_n(x) \quad \left(\text{generating function}\right).$$

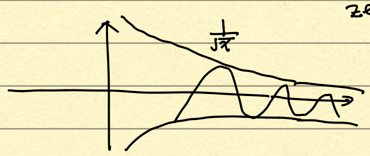
✓ still right.

- Asymptotic Behavior of $J_n(x)$.

as $x\to\infty$

- $$J_n(x) = \underbrace{\sqrt{\frac{2}{\pi x}}}_{\text{decay factor}} \cdot \underbrace{\cos\left(x - \frac{2p+1}{4}\pi\right)}_{\text{oscillation factor.}} + O(x^{-\frac{3}{2}}).$$

spacing between zero $\sim \pi$.



$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^n}{k!\ \Gamma(k+n+1)} \left(\frac{x}{2}\right)^{2k+n} \cdots$$

- Stationary Phase Expansion:

  - recall the Gaussian integral:

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = 1$$

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}. \qquad (\alpha>0).$$

change variables: s.t. $\alpha x^2 = \frac{1}{2}u^2$.

$\Leftrightarrow \quad u = \sqrt{2\alpha}\cdot x, \qquad x = \sqrt{\frac{1}{2\alpha}}\, u$

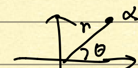
$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}u^2} \cdot d\left(\frac{1}{\sqrt{2\alpha}}\cdot u\right)$$

$$= \frac{1}{\sqrt{2\alpha}} \left(\int_{-\infty}^{+\infty} e^{-\frac{1}{2}u^2} \cdot \frac{du}{\sqrt{2\pi}} \right) \sqrt{2\pi}.$$

$$= \sqrt{\frac{\pi}{\alpha}}.$$

- we can allow α to be a complex number:

say $\alpha = r\cdot e^{i\theta}$ $\qquad \underbrace{-\frac{\pi}{2} < \theta < \frac{\pi}{2}}$



$$\int_{-\infty}^{+\infty} e^{-r\cdot e^{i\theta}\cdot x^2} dx = \sqrt{\frac{\pi}{r\cdot e^{i\theta}}}$$

formula still works.

But, let's study how the integrand behave

$$\alpha = r\cdot e^{i\theta} = r\cdot\cos\theta + i\cdot r\sin\theta = A + iB$$

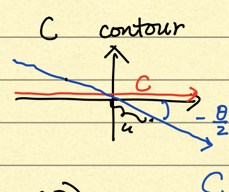
$$\int_{-\infty}^{+\infty} e^{-(A+iB)\cdot x^2} dx$$

$$= \int_{-\infty}^{\infty} \underbrace{e^{-Ax^2}}_{\text{decay factor if } A>0} \cdot \underbrace{e^{-iBx^2}}_{\text{oscillation factor}} dx$$

BAD!

To kill the oscillation factor, we "tilt" the integration contour: allow x to be complex.
rename x to z, we do.

$$I = \int_C e^{-\alpha\cdot z^2} dz$$



rotate the ~~counte~~ contour, so that

$$z = u\cdot e^{-i\frac{\theta}{2}} \qquad (u\in\mathbb{R})$$

$$\alpha \cdot z^2 = (r \cdot e^{i\theta}) \cdot (u \cdot e^{-i\frac{\theta}{2}})^2$$

$$= r \cdot u^2$$

$r > 0$

$u \in \mathbb{R}$

$u^2 \geq 0$

$$I = \int_{C'} e^{-\alpha z^2} dz$$

$$= \int e^{-ru^2} d(u \cdot e^{-i\frac{\theta}{2}})$$

$$= e^{-i\frac{\theta}{2}} \cdot \int_{-\infty}^{+\infty} e^{-ru^2} \cdot du$$

$$= e^{-i\frac{\theta}{2}} \cdot \sqrt{\frac{\pi}{r}}$$

In general: if we are given an integral

$$\int_{-\infty}^{+\infty} e^{\frac{1}{\hbar} \cdot S(x)} \cdot dx$$

and if $S(x)$ has a critical point x_0.

then, we may expand $S(x)$ near x_0.

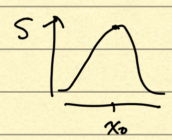
$$S(x) = S(x_0) + \frac{1}{2} S''(x_0) \cdot (x - x_0)^2 + \cdots$$

$$\int_{-\infty}^{+\infty} e^{\frac{1}{\hbar} S(x_0) + \frac{1}{\hbar} \cdot \frac{1}{2} \cdot S''(x_0) \cdot (x - x_0)^2} dx$$

if we let $u = x - x_0$, then.

$$\approx e^{\frac{1}{\hbar} S(x_0)} \cdot \int_{-\infty}^{+\infty} e^{A \cdot \frac{u^2}{\hbar}} \cdot du.$$

$$A = \frac{1}{2} S''(x_0).$$



$$\approx e^{\frac{1}{\hbar} S(x_0)} \cdot \sqrt{\frac{\pi}{-A/\hbar}}$$