Ch 13. Partial Diff Eqn.

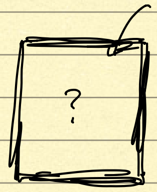
13.1. Diff Eq:

- Heat Eq. : $\partial_t u(t,\vec{x}) = \nabla^2 \cdot u(t,\vec{x})$
- Schroedinger Eq : $i\partial_t u = -\nabla^2 u$
- wave Eq : $\partial_t^2 u = \nabla^2 u$
- Helmholtz Eq : $\lambda \cdot u = \nabla^2 u$

(eigen-value problem for ∇^2).

13.2 Separation of Variable for steady-state heat eq in a rectangle.

- metal plate.

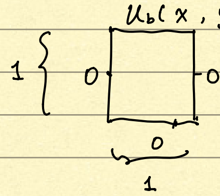
D = 

- specify the temperature on the boundary

Q: what is the temperature inside.

$$\begin{cases} 0 = \nabla^2 u. & \text{on } D \\ u \text{ on the boundary of } D \text{ given.} \end{cases}$$

First solve a simpler problem, setting $u_b(x, y=1)$ given, temperature on 3 sides to be zero.



step 1: write general sol'n satisfying these homogeneous B.C.

$$u(x,y)_n = \sin(n\pi x)\cdot \sinh(n\pi y).$$

indeed,

$$(\partial_x^2 + \partial_y^2)(\sin(n\pi x)\cdot\sinh(n\pi y))$$

$$= \left[-(n\pi)^2 + (n\pi)^2\right]\cdot\left(\sin(n\pi x)\cdot\sinh(n\pi y)\right)$$

$$= 0.$$

general sol'n to the Homogen B.C.

$$u(x,y) = \sum_{n=1}^{\infty} a_n \cdot u_n(x,y)$$

step 2: choose a_n such that the B.C. at the top edge is satisfied.

$$u_b(x,1) = \sum_{n=1}^{\infty} a_n \cdot \sin(n\pi x)\cdot \underline{\sinh(n\pi)}$$

Now we obtain a_n by Fourier expansion of $u_b(x,1) = \sum_{n=1}^{\infty} C_n \cdot \sin(n\pi x)$

$\Rightarrow C_n = a_n \cdot \sinh(n\pi)$

- To solve the general problem with 4 sides non-zero, we just consider 4 sub-problems, each with a side temp non-zero. $\rightsquigarrow$

$$u(x,y) = u_{top}(x,y) + u_{bottom}(x,y) + u_{left}(x,y) + u_{right}(x,y)$$

- $u(x,y)$ satisfy eqn $\nabla^2 u = 0$
- $u(x,y)$ satisfy B.C. of four sides.

- Method of separation of variables to get gen. sol'n for

① $(\partial_x^2 + \partial_y^2) u = 0.$

assume $u(x,y) = X(x)\cdot Y(y).$

② $(\partial_x^2 \cdot X)\cdot Y + X\cdot \partial_y^2(Y) = 0$

divide by $\frac{1}{XY}$

③ $$\frac{1}{X}\partial_x^2 X + \frac{1}{Y}\cdot\partial_y^2 Y = 0$$

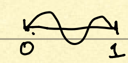
$\Rightarrow \exists\, \lambda$, s.t.

$$\begin{cases} \frac{1}{X}\partial_x^2 X = \lambda \\ \frac{1}{Y}\partial_y^2 Y = -\lambda. \end{cases}$$

④ consider the Boundary condition:

$X(0) = 0,\ X(1) = 0.$

$\partial_x^2 X = \lambda X$ for some λ.



$X_n(x) = \sin(n\pi\cdot x)$ solves the eqn.

with $\lambda_n = -(n\pi)^2$.

remark: (1) we want to consider X eqn first, because it has 2 Boundary conditions $\Rightarrow$ eigenvalues are discrete.

$$\lambda_n = -(n\pi)^2.$$

(2) if we consider Y-eqn first.

$$\begin{cases} \partial_y^2 Y(y) = -\lambda\cdot Y & \text{on } y\in[0,1]. \\ Y(0) = 0 \end{cases}$$

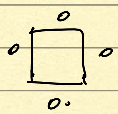
λ can be any real number

$Y(y) = \sin(\sqrt{\lambda}\cdot y)$. always solve the eqn + BC.

so it does not narrow down our selection of eigenvalues.

13.3. Heat-Diffusion on a square:

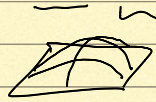
$$\begin{cases} \partial_t u(t,x,y) = \partial_x^2 u(x,y) + \partial_y^2 u(x,y). \quad ① \\ \qquad x \in [0,1], \quad y \in [0,1]. \\ u(0,x,y) = u_0(x,y). \quad ② \\ u(t,0,y) = u(t,1,y) = u(t,x,0) \quad ③ \\ \qquad = u(t,x,1) = 0 \end{cases}$$



step 1: Consider the homogeneous problem. ① + ③. , try to find the gen sol'n.

by ③. $u_{n,m}(t,x,y) = \underline{T(t)} \cdot \sin(n\pi x) \cdot \sin(m\pi y)$ satisfies ③.

ex. $u_{1,1}(t,x,y) = \underline{T(t)} \cdot \underbrace{\sin(\pi x)\sin(\pi y)}$.



$u_{n,m} = T \cdot X \cdot Y$

using ①, we see.

$$\partial_t u_{n,m}(t,x,y) = (\partial_x^2 + \partial_y^2) \cdot u_{n,m}(t,x,y)$$

$$\frac{1}{T}(\partial_t \cdot T) = \frac{1}{X}\partial_x^2 \cdot X + \frac{1}{Y}\partial_y^2 \cdot Y.$$

$$= -(n\pi)^2 - (m\pi)^2.$$

$$\Rightarrow \quad T(t) = c \cdot e^{-[(n\pi)^2 + (m\pi)^2] \cdot t}$$

Then the gen sol'n to ① + ③ is

$$u(t,x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \underline{a_{nm}} \cdot e^{-[(n\pi)^2 + (m\pi)^2]t} \cdot \sin(n\pi x) \cdot \sin(m\pi y).$$

step 2: to adjust $a_{n,m}$ so that the B.C. ② is satisfied.

$$u_0(x,y) = \sum_{n,m} a_{n,m} \sin(n\pi x) \cdot \sin(m\pi y).$$

To get a_{n_0, m_0} coeff, we apply the following to the eq.

$$\int (\cdots) \cdot \underline{\sin(n_0 \pi x)} \cdot \underline{\sin(m_0 \pi y)} \cdot dx\,dy$$

$$LHS = \int_0^1\int_0^1 \underline{u_0(x,y)} \cdot \sin(n_0\pi x)\sin(m_0\pi y)\,dx\,dy$$

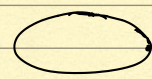
$$RHS = a_{n_0 m_0} \cdot \int \sin(n_0\pi x)^2 \cdot \sin^2(m_0\pi y)\,dx\,dy$$

$$= a_{n_0 m_0} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

LHS = RHS $\Rightarrow$ $a_{n_0 m_0} = \cdots$

Ex: Consider the example of heat diffusion on circle S^1.

$$\begin{cases} \partial_t u = \partial_\theta^2 u \qquad u(t,\theta) \\ u(0,\theta) \text{ given.} \end{cases}$$



find eigen fcn of ∂_θ^2 on S^1.

$$u(t,\theta) = T(t) \cdot \Theta(\theta).$$

$$\begin{cases} \partial_\theta^2 \Theta = \lambda \cdot \Theta \quad ① \text{ — solve this first.} \\ \partial_t T = -\lambda \cdot T \end{cases}$$

eigenvalue problem for ①. $\Rightarrow$

$\Theta = \sin(n\theta)$. $\quad n \geq 1$. integer

or

$\Theta = \cos(n\theta)$. $\quad n \geq 0$.

eigenvalue: $\lambda_0 = 0 \qquad \Theta = 1.$

$\lambda_1 = -1^2, \qquad \Theta = \sin(\theta), \cos(\theta).$

$\vdots$

$\lambda_n = -n^2. \qquad \Theta = \sin(n\theta), \cos(n\theta)$

Read online for the sol'n....