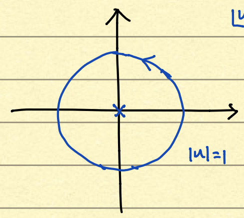
(1) Generating function for Bessel fcn: $J_n(z)$.

$$e^{\frac{x}{2}(u-\frac{1}{u})} = \sum_{n=-\infty}^{+\infty} u^n \cdot J_n(x) \quad \checkmark$$

$$J_n(x) = \oint_{|u|=1} u^{-1-n} \cdot e^{\frac{x}{2}(u-\frac{1}{u})} \cdot \frac{du}{2\pi i} \quad \checkmark$$

$$\neq \frac{1}{n!}\left(\frac{d}{du}\right)^n \Big|_{u=0} \left(e^{\frac{x}{2}(u-\frac{1}{u})} \right) \quad \times$$



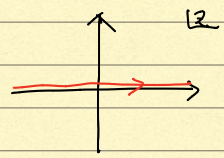
Q: what happens if $x \gg 1$?
can we find simple approximation for $J_n(x)$.

By Steepest Descent method:

contour
an integral $\displaystyle\int_C f(z)\, e^{\lambda \cdot S(z)}\, dz$

for $\lambda \gg 1$. will receive its main contribution near critical points of $S(z)$.

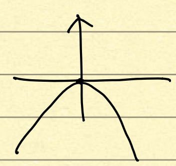
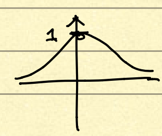
Example for SD method:

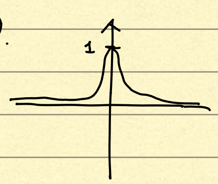


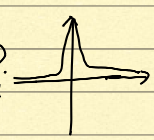
$$\int_{\mathbb{R}} e^{\lambda\cdot(-\frac{1}{2}z^2)} \cdot dz$$

$$S(z) = -\tfrac{1}{2}z^2.$$

$S'(z) = -z$, crit pt is where $S'(z)=0$
$\Rightarrow z=0.$

$S(z)$:  , $e^{S(z)}$: 

$e^{\lambda S(z)}$. 

, $\sqrt{\lambda}\cdot e^{\lambda S(z)}$: 

$$\int e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}.$$

$$\int e^{\lambda(-\frac{1}{2}z^2)} \underset{\sim}{dz} = \frac{1}{\underset{\sim}{\sqrt{\lambda}}}\cdot\sqrt{2\pi}.$$

change variable. Let $\sqrt{\lambda}\cdot z = x.$, $z = \frac{x}{\sqrt{\lambda}}$

$$\int z\cdot e^{-\lambda\cdot\frac{1}{2}z^2} dz = 0 \qquad \text{(by symmetry reason).}$$

$$\int_{-\infty}^{+\infty} \underset{\sim}{z^2}\cdot e^{-\lambda\cdot\frac{1}{2}z^2} \underset{\sim}{dz} =$$

replace $\sqrt{\lambda}\, z$ by x. $z = \frac{x}{\sqrt{\lambda}}.$

$$= \int \left(\frac{x}{\sqrt{\lambda}}\right)^2 \cdot e^{-\frac{1}{2}x^2} \cdot d\left(\frac{x}{\sqrt{\lambda}}\right).$$

$$= \left(\frac{1}{\sqrt{\lambda}}\right)^3 \cdot \int_{-\infty}^{+\infty} x^2 e^{-\frac{1}{2}x^2}\cdot dx = \lambda^{-\frac{3}{2}}\, C$$

$$\int 1\cdot e^{-\frac{1}{2}\lambda z^2}\cdot dz = \lambda^{-\frac{1}{2}}$$

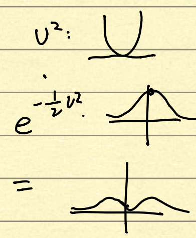
In general, if we have a critical point of $S(z)$ at z_0 $\qquad z - z_0 = v$

$$\int_{-\infty}^{+\infty} f(z)\cdot e^{\lambda\cdot S(z)} dz \approx \int_{-\infty}^{+\infty} \left[f(z_0) + f'(z_0)\cdot \underline{v} + f''(z_0)\frac{v^2}{2} \right] \cdot e^{\lambda(S(z_0) + \frac{1}{2}S''(z_0)\cdot v^2)} \cdot dv.$$

say $f(z_0) \neq 0$.

$$= f(z_0)\cdot e^{\lambda\cdot S(z_0)} \cdot \int_{-\infty}^{+\infty} \Big(1 + \underbrace{\frac{f'(z_0)}{f(z_0)}\cdot v}_{\text{approx}=0} + \underbrace{\frac{f''(z_0)}{f(z_0)}\cdot\frac{v^2}{2}}_{\text{approx}=0}\Big) \cdot \underbrace{e^{\lambda\cdot \frac{1}{2}S''(z_0)\cdot v^2}}\cdot dv.$$

$$= f(z_0)\cdot e^{\lambda S(z_0)} \cdot \sqrt{\frac{2\pi}{-\lambda\cdot S''(z_0)}}$$

v^2: , $e^{-\frac{1}{2}v^2}$: , =


we are choosing contour of v, such that

- $\frac{1}{2}S''(z_0)\cdot v^2$ is real
- and it is <u>decreasing along</u> both directions of the contour.

Ex: $\cdot\ \frac{1}{2}S''(z_0)\cdot v^2 = i\cdot v^2$

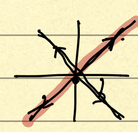
then. to make it real, say $v = r\cdot e^{i\theta}$,

$i\cdot r^2\cdot e^{2i\theta}$ is real.

$= r^2\cdot e^{2i\theta + i\frac{\pi}{2}}$ is real

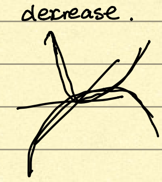
$\underline{2\theta} + \frac{\pi}{2} = 0$ or π (mod 2π).

$\theta = -\frac{\pi}{4}$



- if $v = r\cdot e^{-i\frac{\pi}{4}}$,
  $iv^2 = i\cdot r^2\cdot e^{-i\frac{\pi}{2}} = r^2$ BAD, since it increases as $r\to\infty$
- if $v = r\cdot e^{i\frac{\pi}{4}}$
  $i\cdot v^2 = \underline{i}\cdot r^2\cdot e^{i\frac{\pi}{2}} = -r^2.$

then iv^2 will decrease. along this direction.



Now, to the Bessel function:

$$J_n(x) = \oint_{|u|=1} u^{-1-n} \cdot e^{\frac{x}{2}(u-\frac{1}{u})} \cdot \frac{du}{2\pi i}$$

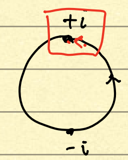
let $S(u) = u - \frac{1}{u}$, then

$S'(u) = 1 + \frac{1}{u^2}$, $\quad S''(u) = \frac{-2}{u^3}$

so crit pts are obtained by solving

$S'(u) = 0$, $\quad 1 + \frac{1}{u^2} = 0 \Rightarrow u^2 = -1 \Rightarrow u = \pm i$

- these critical points are already on the contour, i.e. $|u| = 1$.



- we still need to make sure the contour passes through crit pts in the "steepest descent direction".

Near $u_0 = i$, let $u - u_0 = v$, $u = u_0 + v$.
the integral has the following form

$$I_1 = \int (u_0+v)^{-1-n} \cdot e^{\frac{x}{2}(u_0+v-\frac{1}{u_0+v})} \frac{d(u_0+v)}{2\pi i}$$

$$\approx \int_{+\infty}^{-\infty} u_0^{-1-n} \cdot e^{\frac{x}{2}[S(u_0)+\frac{1}{2}S''(u_0)\cdot v^2]} \cdot \frac{dv}{2\pi i}$$

$S(u_0) = S(i) = i - \frac{1}{i} = 2i$ $\qquad i^4 = 1$

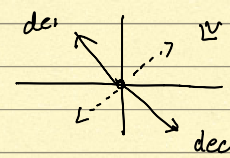
$S''(u_0) = -\frac{2}{u_0^3}\Big| = -\frac{2}{i^3} = -2i$

$$= \int_{-\infty}^{+\infty} (-1)\cdot i^{-1-n} \cdot e^{\frac{x}{2}((2i)+\frac{1}{2}(-2i)\cdot v^2)} \cdot \frac{dv}{2\pi i}$$

$$= \frac{(-1)\cdot i^{-1-n} \cdot e^{\frac{x}{2}(2i)}}{2\pi i} \cdot \int_{-\infty}^{+\infty} e^{-\frac{xi}{2}\cdot v^2} \cdot dv^2.$$

we need $(-i\cdot v^2)$ to be
① real.
② decreasing along the contour

$\Rightarrow v = r\cdot e^{-i\frac{\pi}{4}}$ and $v = r\cdot e^{i\frac{3\pi}{4}}$ $(r>0)$.



$v = r\cdot e^{-i\frac{\pi}{4}}$, let r go from $-\infty, +\infty$, $r\in\mathbb{R}$

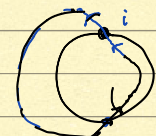
$$\int_{-\infty}^{+\infty} e^{-\frac{x}{2}\cdot i v^2} \cdot dv$$

$$= \int_{-\infty}^{+\infty} e^{-\frac{x}{2}r^2} \cdot d(r\cdot \boxed{e^{-i\frac{\pi}{4}}})$$

$$I_1 = \frac{(-1)\cdot i^{-1-n}\cdot e^{\frac{x}{2}\cdot(2i)}}{2\pi i} \cdot e^{\frac{\pi}{4}i} \cdot \int_{-\infty}^{+\infty} e^{-\frac{x}{2}\cdot r^2} \cdot dr$$

$$= e^{ix-(n+1)\frac{\pi}{2}i+\pi i+\frac{\pi}{4}i} \cdot (2\pi i)^{-1} \cdot \sqrt{\frac{2\pi}{x}}$$

contribution for integral near i.



Similarly, near $u = -i$,
the contour needs to go into $-i$, $I_2 =$

Then $J_n(x) = I_1 + I_2 = \cdots$ see online note.

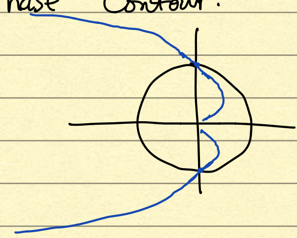
constant phase means.

can be any complex $\qquad$ real

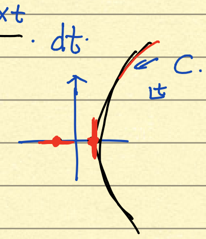
$$e^{\lambda\cdot S(z)} \approx e^{\lambda S(z_0) + (\lambda S''(z_0)\cdot v^2)}$$

constant phase $\Rightarrow$ $\operatorname{Im} S(z) = \operatorname{Im} S(z_0)$
along the contr

Constant Phase Contour:



$$Ai(x) = \int_C e^{t^3 - xt} \cdot dt.$$



say. $\qquad 3t^2 - x = 0$

$t = \pm\sqrt{x}$ $\qquad x > 0$.